Algebra-II

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Lecture 49

Initial and Terminal Objects

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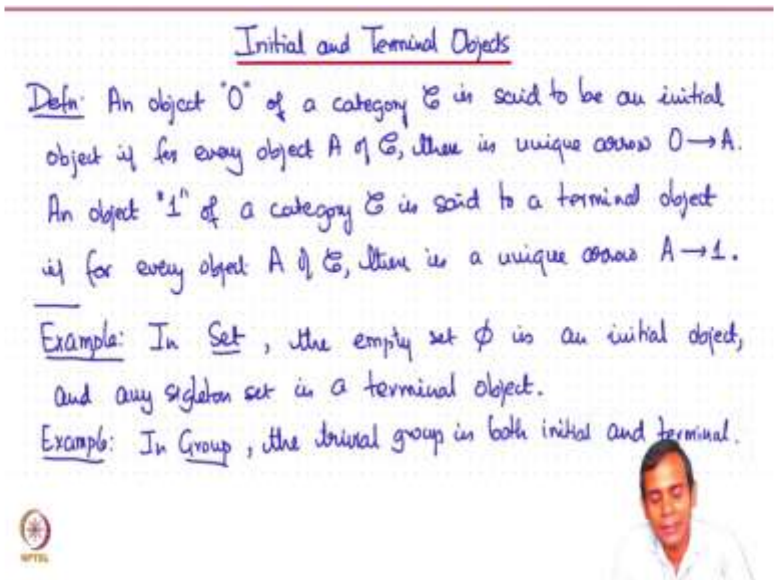

In this lecture we will study Initial and Terminal Objects. This is just a warm up to learning about universal properties that are used to define a lot of constructions in various categories such as products, direct sums, quotients, etc. So, the definition of an initial object is very simple. It is almost so simple that it does not look like it could be anything interesting, but here it is.

An object 0, so usually we denote initial objects by 0 of a category C is said to be an initial object if for every object A of C there is a unique arrow from 0 to A. And while we are at it let us also define a terminal object it is the dual concept in the sense that a terminal object in a category C is an initial object in the category C opp. So, you can guess what the definition would be.

An object, okay and now the notation usually for terminal object is 1 of a category C is said to be a final terminal object if for every object A of C there is a unique arrow A to 1. So, whenever we see a definition in category theory it is good to ask what does it mean for sets. So, let us look at the category of sets. So, in the category Set what is an initial object?

So, you are looking for a set such that given any set there is a unique function from this set, this initial set to any set and the only set that qualifies seems to be the empty set, is an initial object. In fact, it is the only initial object. What is that function from the empty set to any set it is the empty function. Its domain is empty so you do not really need to define its values at all, and that is the empty function and what is the final object?

Well in this case it turn out that there are many final objects. Every singleton set is a final object, because if you have a singleton set then from any other set there is exactly one arrow from any set. There is exactly one function from any set to a singleton set and, so any singleton set is a final object, it is a terminal object. So, the initial object and terminal object are different in the category of sets. Let us look at another example - The category of groups.

So, we have this group called the trivial group, which has only one element namely the identity element and that is an initial object and a final object. Well, if you have the trivial group it has only an identity element to define a homomorphism from a trivial group to any group you only need to define it at the identity and well a group homomorphism must take the identity to the identity so in this sense groups are a little different from sets so the identity element has to go to the identity so there is only one arrow from the trivial group to any group.

And if you take any group you have only one homomorphism to the trivial group because every element has to just map to the identity and that turns out to be a homomorphism. So, the trivial group is also a terminal object. So, here in this category group the initial and terminal object, the trivial group is both initial and terminal.

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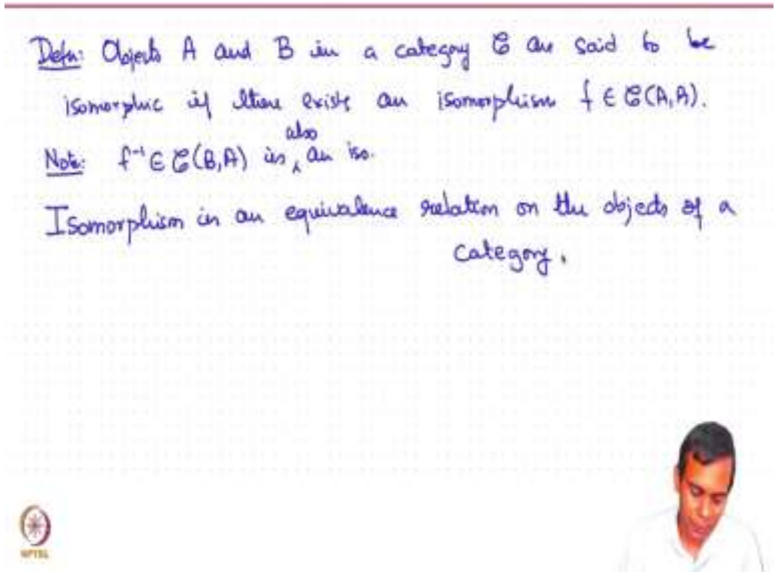

So, it turns out that there are many different, there could be many different initial objects or there could be many different terminal objects in a category but turns out that they are all isomorphic. So, firstly, let me talk about what an isomorphism in a category is, two objects A and B in a category C are said to be isomorphic if there exists an isomorphism f from A to B.

And note that f inverse then is an isomorphism from B to A, so it is also an isomorphism. So, this is a symmetric relationship in fact it is an equivalence relation. So, isomorphism is an equivalence relation on the objects of a category. We have seen that in the category of groups isomorphism preserves all the group theoretic properties of a group in the category of modules, isomorphism preserves all the module theoretic properties of the module, namely properties which can be formulated purely in terms of it being a, its module operations.

So, isomorphism, but this is a more abstract notion of isomorphism because the objects A and B need not even be sets they could just be objects in some weird category, but still intuitively we are saying that isomorphic objects in a category would have the same properties and can be regarded as the same, however there could be many different isomorphisms between two objects. Now, initial and terminal objects have a very special property, so let me state it.

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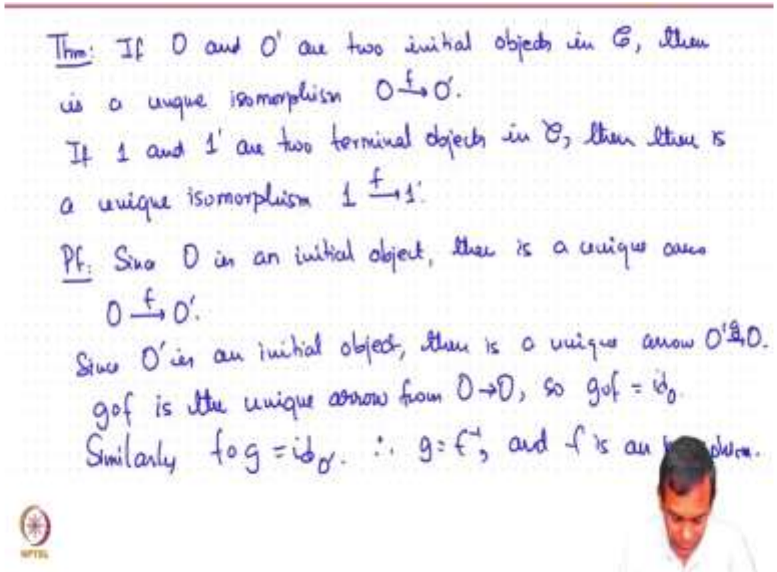

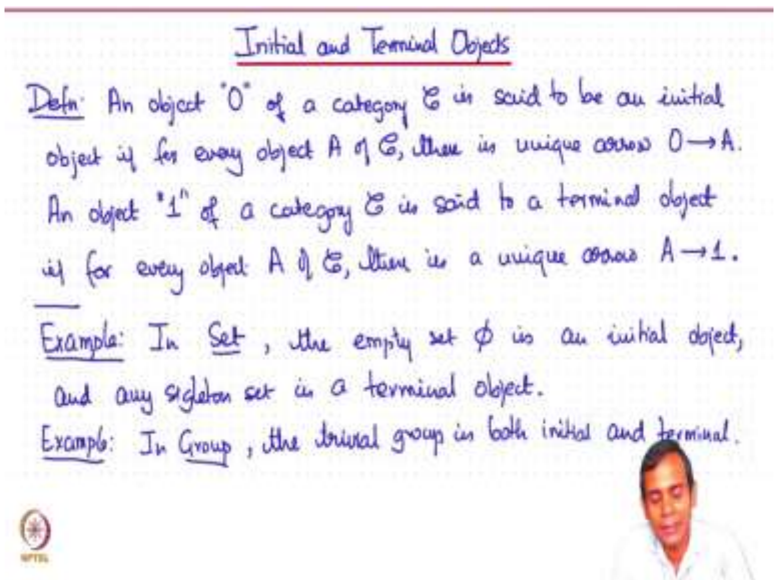

Theorem, if 0 and 0 prime are two initial objects in C then there is a unique isomorphism from 0 to 0 prime and similarly it is true of terminal objects if 1 and 1 prime are two terminal objects in C then there is a unique isomorphism, and this theorem is almost, is completely trivial to prove because if, since 0 is an initial object there is a unique, so let us prove this.

Let us prove the first assumption. Since, 0 is an initial object there is a unique arrow f from 0 to 0 prime and since 0 prime is an initial object there is a unique arrow g from 0 prime to 0. Now, I want to claim that g is the inverse of f. Now g circle f is an arrow from where? So, f goes from 0 to 0 prime and g goes from 0 prime to 0, so g circle f is from 0 to 0. So, g circle f is the unique arrow from 0 to 0.

There is a unique arrow from 0 to 0 just because 0 is an initial object. So, this g circle f is the unique arrow from 0 to 0 and hence it has to be the arrow, identity of 0, so because 0 to 0 there must be an identity arrow so g circle f is identity of 0 and similarly f circle g is identity of 0 prime, therefore g is the inverse of f and f is an isomorphism.

So, even though there maybe multiple initial objects in a category they are all isomorphic and moreover there is a unique isomorphism between them and the same is true of terminal objects. The proof is very similar. I leave it for you to write it down and, so for example, in the category of set singleton set is a terminal object.

There are many singleton sets but all singleton sets are isomorphic in the category of sets and there is a unique isomorphism from any singleton set to any other singleton set.