Algebra-II

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Lecture 48

Categories: First Problem Session

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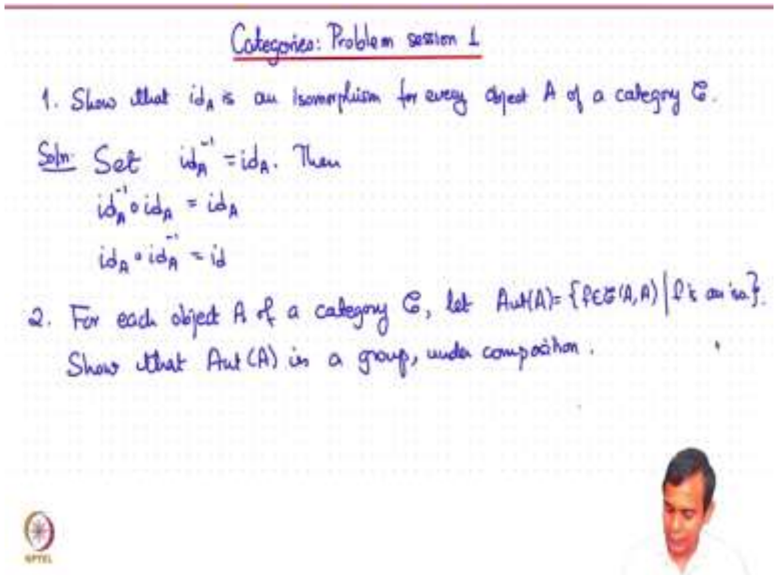

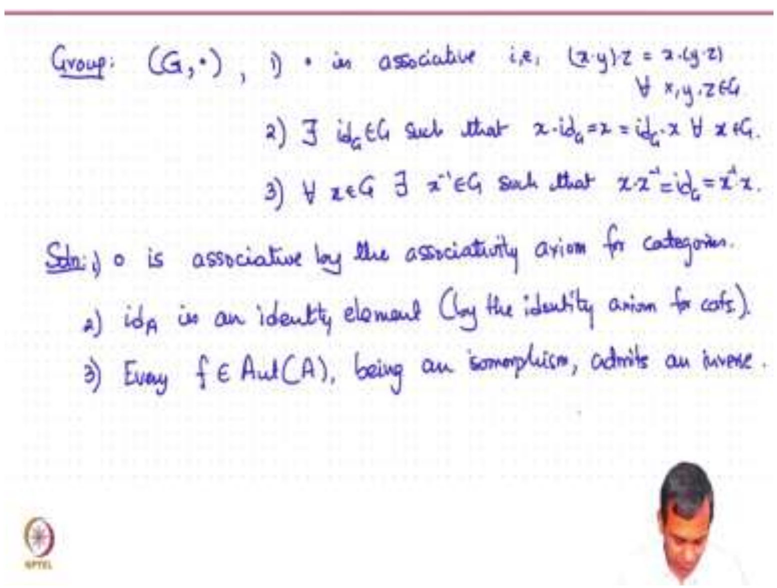

So, let us reinforce our understanding of categories by solving some problems. The first problem show that the identity arrow is an isomorphism for every object A of category C. So, recall the definition of an isomorphism and arrows called an isomorphism if you can find an inverse arrow.

So, identity A inverse is equal to identity A and in fact we see that this is the case because identity A inverse which is identity A circle identity A is equal to identity A and so if you treat this as identity A inverse then we are getting this and we also get identity A circle identity inverse equal to identity A. Both these come from the identity axiom.

So, just to write this down nicely, I will say Set identity A inverse equal to identity A, then we have these things which establishes that identity A is the inverse of the identity. So, that was easy, let us try something a little more interesting. So, for each object A of a category C, let Aut A equals f arrows from A to A such that f is an isomorphism.

Show that Aut A is a group. So, before we go in to this let us recall the definition of a group. It is a set G together with an operation, a binary operation dot, such that dot is associative i e, x dot y dot z is equal to x dot y dot z for all x y z in G, then for every object, for every element x in G, oh no but first I should state the identity axiom, there exists identity a in G, such that x dot identity a is equal to x and that is also equal to identity dot x for all x in G.

And the third axiom is axiom of the inverse, so there is associativity, identity and inverse axiom. For every x in G there exist x inverse in G such that x dot x inverse is equal to identity a and that is also equal to x inverse dot x. So, these are the axioms for a group and we have to show is that this set of arrows is a group. Let us say with the binary operation under composition. So, let me just make that clear. And, so we just need to check these axioms.

Now, the solution, of course composition is associative by the associativity axiom for categories, and then there exists an identity which is identity sub A by the identity axiom for categories, and thirdly the existence of an inverse. Well, that comes from the fact that we have defined it to be a set of isomorphisms. Every f being an isomorphism admits an inverse. So, that is it, the two axioms of associativity and identity follow from the axioms of a category and the third axiom of inverse just comes from this definition of Aut A.

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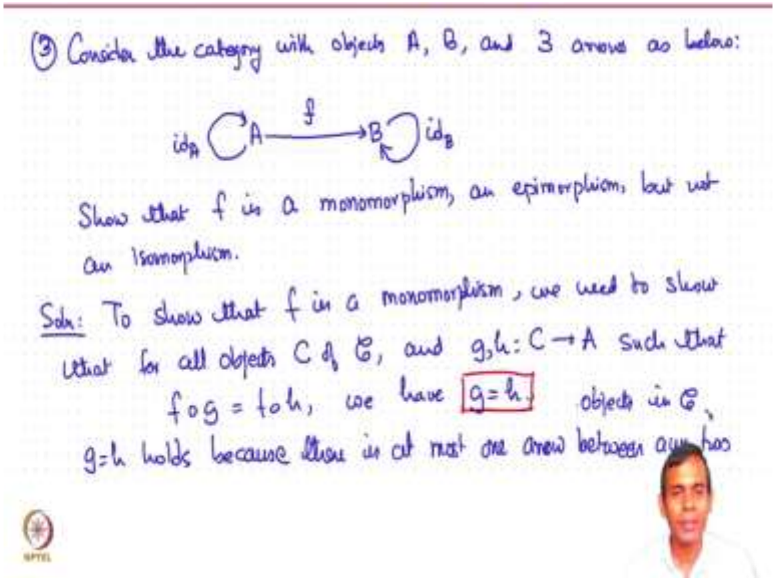

The third problem – Consider the category with two objects which are called A and B and 3 arrows as below. So, let me show you what the arrows look like. So, we have an object A, we have an object B. We must have an arrow identity A from A to A, we must have an arrow identity B from B to B and now there is only one, we will have let us say an arrow f from A to B.

So, show that f is a monomorphism and epimorphism but not an isomorphism. So, just pause your video and think about this. You need to maybe go look up the definitions of monomorphism and epimorphism and isomorphism and then try to solve this problem. If not you can watch me explain the solution. This is just a matter of a definition.

So, recall what it means for f to be a monomorphism. So, f is a monomorphism means that, to show that f is a monomorphism we need to show that for all objects C of the category C and arrows let us say g and h from C to A, such that f circle g equals f circle h. We have g is equal to h. But in this category there is only one morphism between any two objects.

There is only morphism from A to A, there is only morphism from B to B, there is only one morphism from A to B and actually there are no morphisms from B to A, so there is at most one morphism between any two objects. And so this conclusion here is trivially satisfied. So, no matter what kind of composition you have, so there is only one morphism from any object C to A the g equal to h is vacuously true.

So, g equals h holds because there is at most one arrow between any two objects in C. So, that shows that f is a monomorphism and the same argument also shows that f is an epimorphism. Now is f an isomorphism?

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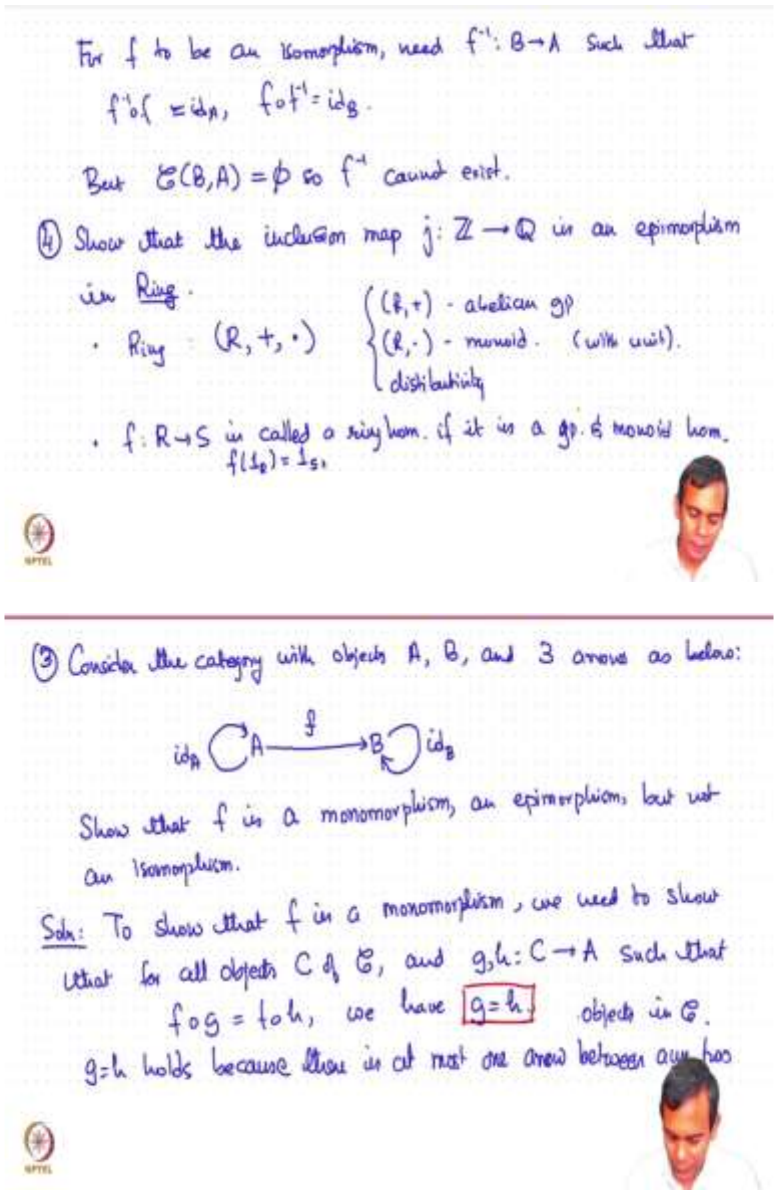

So, recall the definition of isomorphism for f to be an isomorphism we would need f inverse from B to A such that f inverse circle f is the identity of A and f circle f inverse is the identity of B, but do not even worry about these conditions, there is no arrow from B to A. So, f inverse cannot exist and so that means that f is not an isomorphism.

So, this again runs contrary to our intuition from set theory that if a function is injective and surjective then by definition it is bijective and so an arrow is an isomorphism if and only if it is an epimorphism and a monomorphism, but here we have an arrow that is both as epimorphism and a monomorphism but not an isomorphism.

Problem four, show that the inclusion map j from Z to Q is an epimorphism in the category of rings. Now before solving this problem, let us just recall a few things about rings. So, recall that ring consists of two binary operations, addition and multiplication and we have that ring under addition is in abelian group, ring under multiplication is a monoid.

In particular we are assuming that rings have a multiplicative unit, so rings have a unit, so it is the identity of this monoid is the multiplicative unit, and then these two operations talk to each other via distributivity. This is a ring and so the objects in the category of rings are, in the category ring are rings and morphisms are ring homomorphisms.

So, a ring homomorphism if it is a group homomorphism from R plus to S plus and a monoid homomorphism from R dot to S dot. I am just writing this down somewhat informally but an important point is that F takes the multiplicative unit of R to the multiplicative unit of S. So, this is a very important assumption that we make when we define the category of rings. So, now you can pause your video and try to solve this problem. It is actually quite interesting.

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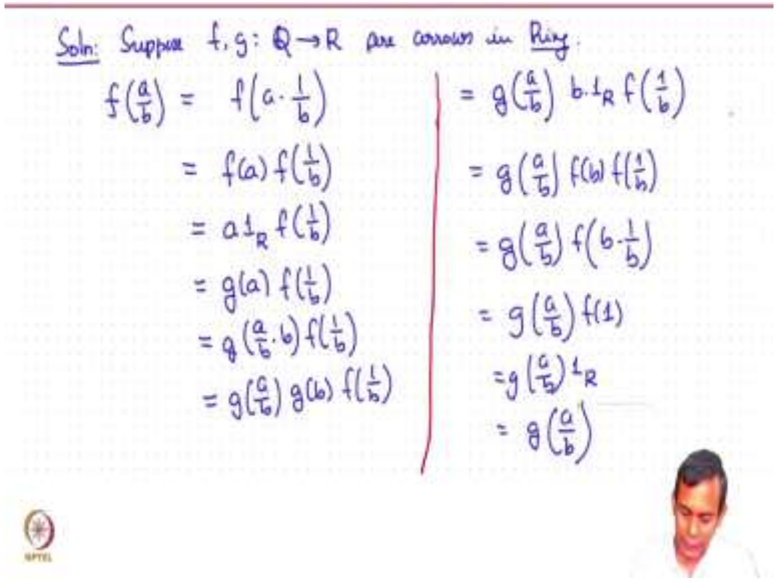

So, here is what I am going to prove. So, suppose f and g are homomorphisms from Q to R, then I claim that f is just equal to (())(15:06), so let us just see so f of a rational number a by b is equal to, so a by b is a times 1 by b and now because f is a homomorphism of the multiplicative monoid what this means is this is f of a times f of 1 over b.

But a here is an integer if a is a positive integer then a is 1 plus 1 plus 1 plus 1 and so f of a is a times f of 1 but f of 1 is 1 of R. So, and if a is negative then well it is the inverse so you can again write this that f of a is a times 1 of R. This is a times 1 sub R, f of 1 over b and then I can write this as, well for the same reason this is equal to g of a there is no difference between f and a so if f of a is supposed to a times 1 of R g of a is also going to be a times 1 of R.

And f of 1 over b, now this is equal to g of a over b times b, f of 1 over b, and now I can write this as g of a by b times g of b times f of 1 over b. Let us just draw a two column format here and move over to the other side so now this is equal to, now g of b as we have seen before this is b times 1 sub R, so this is g of a by b b times 1 sub R f of 1 over b.

And so I can write this as g of a by b and then this is f of b times f of 1 over b, and that is g of a by b, f of b into 1 by b, which is g of a by b into f of 1, which is g of a by b into 1 sub R, which is just g times a by b. So, what we see is that there is only one ring homomorphism from Q to any ring R. They are all equal, and so now automatically this means that j is an epimorphism.

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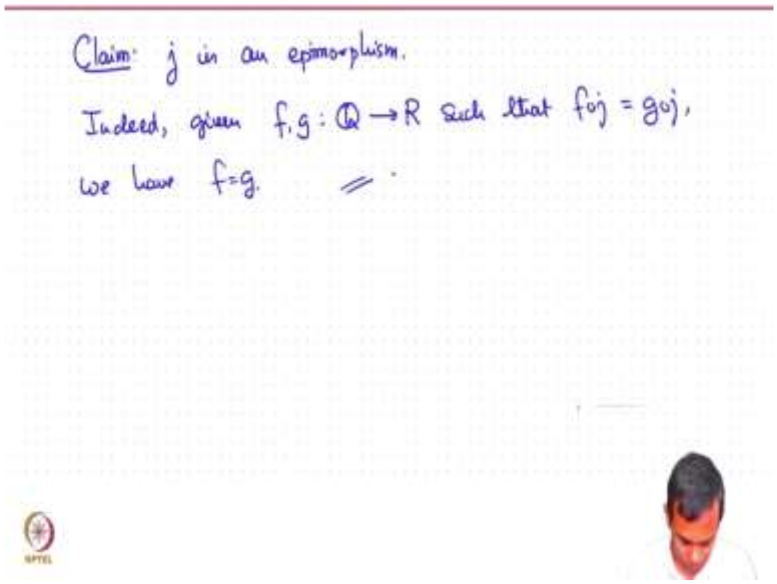

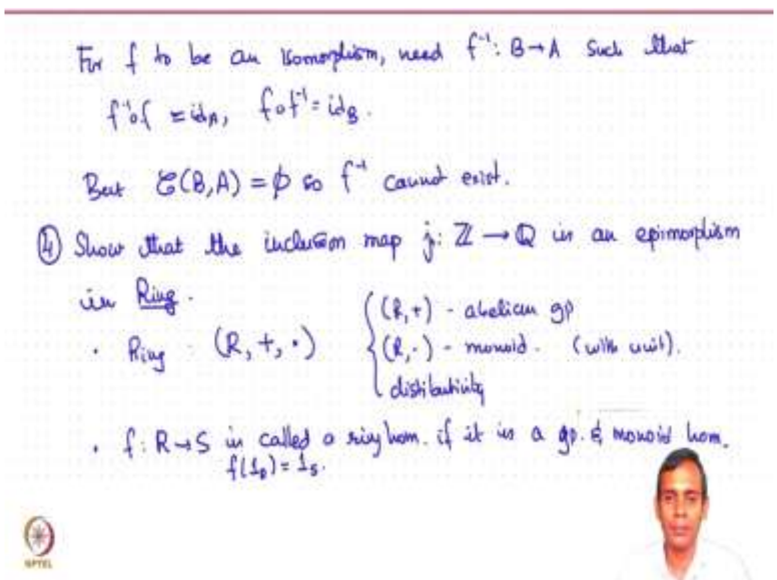

Indeed, given g and, let us call them f and g from Q to R such that f circle J is equal to g circle j, but even without this condition we have f equals g. Because we have f equal g without even this condition and so that finishes the solution and this is very interesting because j is not a surjective function and yet this is an epimorphism.

In fact, j is an epimorphism and a monomorphism because there is an injective function it has to be a monomorphism and we just showed that it is an epimorphism, but it is not an isomorphism.

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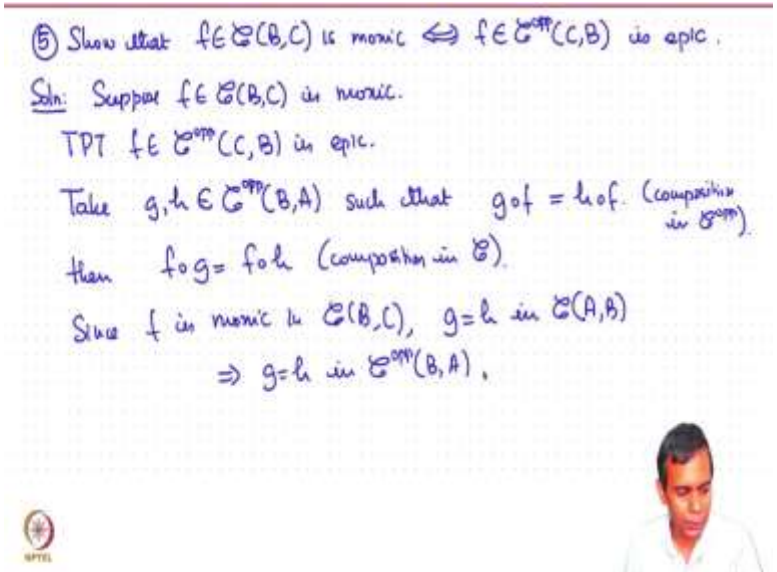

The next problem explores the certain duality that you will see throughout category theory. Each definition and category theory has a dual and basically the dual of a definition is the same definition in the opposite category. So, just to make that explicit let me write down this problem. So, show that f an arrow from B to C in a category is monic if and only if f now can be regarded also as an arrow from C to B in the opposite category.

We call them the opposite categories, the categories with the same objects but the arrows from B to C are thought of as arrows from C to B in the opposite category. So, you should try to just unwind the definition and solve it yourself otherwise just watch me solve it. So, let us just prove, firstly suppose f is monic and I want to prove that this is epic.

So, take any two arrows g and h from B to A and consider such that we have that f circle, g circle f is equal to h circle f and this is the composition in C opp. Then f circle g is equal to f circle h this is composition in C and since f is monic in C B C what we have is g is equal to h in C A B, which is the same as saying that g is equal to h in C opp B A, and therefore f is epic.

Other statement can be done given a similar proof that if f in C opp is epic then f in C B C is monic and I will leave that for you to work out if you feel like.