Algebra-II
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Monomorphisms, epimorphisms and isomorphisms

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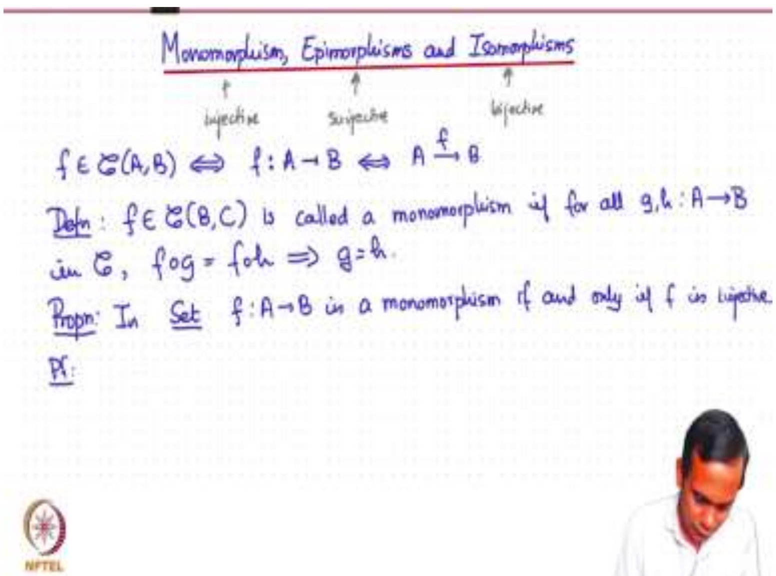

In this lecture I am going to introduce some special kinds of arrows in a category. Now arrows in a category are often called morphisms and these are special kinds of morphisms and they are called monomorphisms, epimorphisms and isomorphisms. So, in the category of sets it will turn out that monomorphisms are injective functions, epimorphisms are surjective functions and isomorphisms are bijective functions.

But that is only for the category of sets. In general these have very specific meanings. So, let us move on to the definitions. Now, one bit of notation I should say that if I have f an arrow in the category C from A to B, I will use this, I will also use the notation f colon A to B, as if f whether it will be a function from the set A to the set B and I will also use the notation A to B, f is an arrow from A to B.

So, in these two notations the category C is not mentioned, it should be implicit. So, now on to the definition of a monomorphism. So, an arrow f in a category, from an object B to an object C is called a monomorphism if for all objects A to B, A in the category C and arrows, let us say g, h from A to B in C. So, when I say for all, I mean for all objects A and all arrows g and h from A to B. We have that f circle g equals f circle h implies that g is equal to h.

So, this is in equality in arrows from A to C and this is in equality of arrows from A to B. Just to show you that this corresponds to injections. So, here is the proposition that in the category Set. So, recall that this is the category whose objects are sets and whose arrows are functions. f A to B is a monomorphism, so a function, this would be a function from a set A to a set B, is a monomorphism if and only if f is injective.

The proof is not very difficult but I will do it carefully nevertheless. So, the thing is that as you will see these things can be a little dicey in category theory. You need to actually prove things so we will prove it carefully. So, maybe I will just start on a fresh page.

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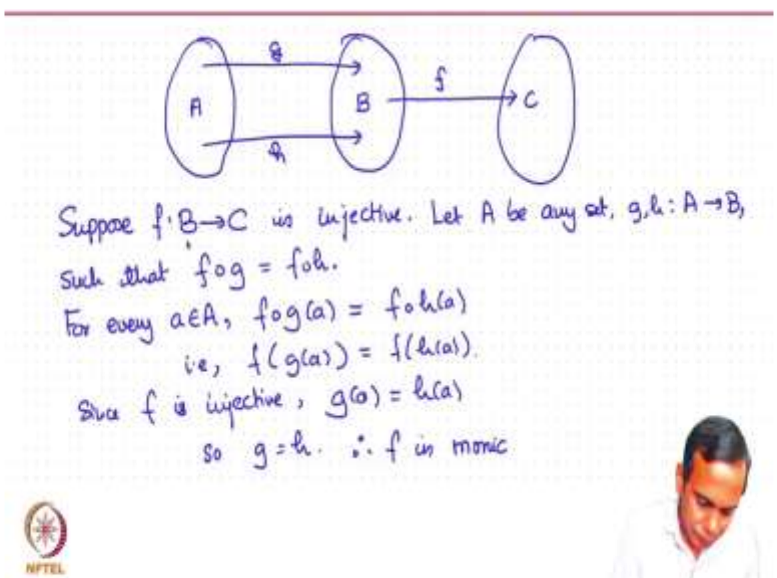

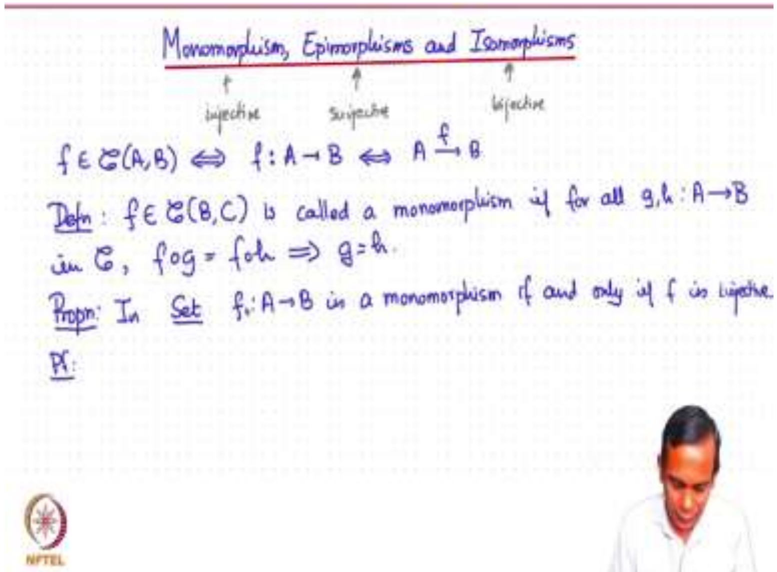

So, the situation can be pictorially denoted like this. We have set A, we have a set B and we have a set C and we have this morphism or a function from B to C, and we have two functions, g and h from A to B. And we need to show that if f circle g is equal to f circle h then for all g and h then f is injective and conversely if f is injective therefore all g and h.

So, now let us start with suppose f is injective. So, maybe I will just say f and remind you that f is from B to C, is injective. Let A be any set and g, h be functions from A to B. Now suppose we know that. So, let g and h be functions such that so the hypothesis, right, such that f circle g is equal to f circle h. Now we need to show that f is equal to g. So, what we have is that for every element a in A, f circle g of a is equal to f circle h of a.

But this is the same as f of g of a is equal to f of h of a, but since f is injective this implies that g of a is equal to h of a and so this synthesis is true for every a in A, g is equal to h. So, this shows that if f is injective then f is a monomorphism. So, f is a monomorphism, that is very long so we have abbreviation. F is monic, so this is a short way of writing f is a monomorphism.

And now we have proved one way. Let us prove the converse. So, for the converse we need to start with this situation and show that F is injective but instead what we will do, we will prove the contra positive. We will show that if the f is not injective then you can construct, you can find a set A and construct functions g and h from A to B, such that g is not equal to h but f circle g is equal to f circle h.

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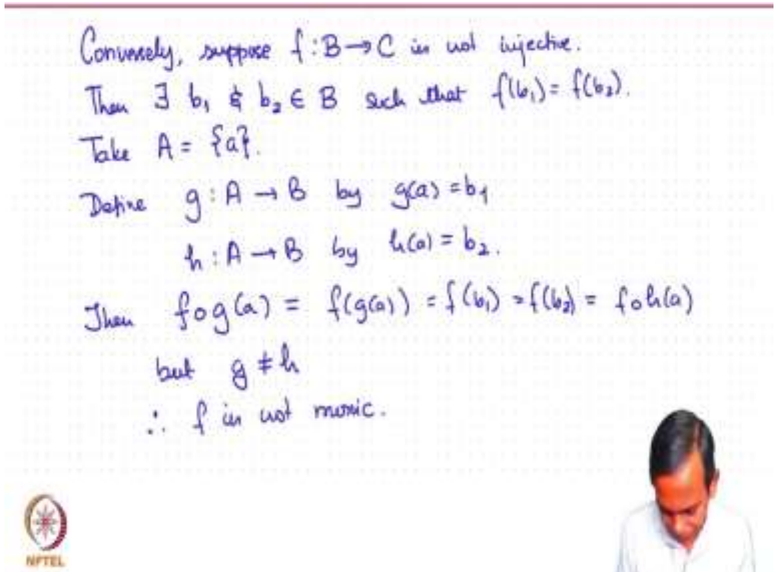

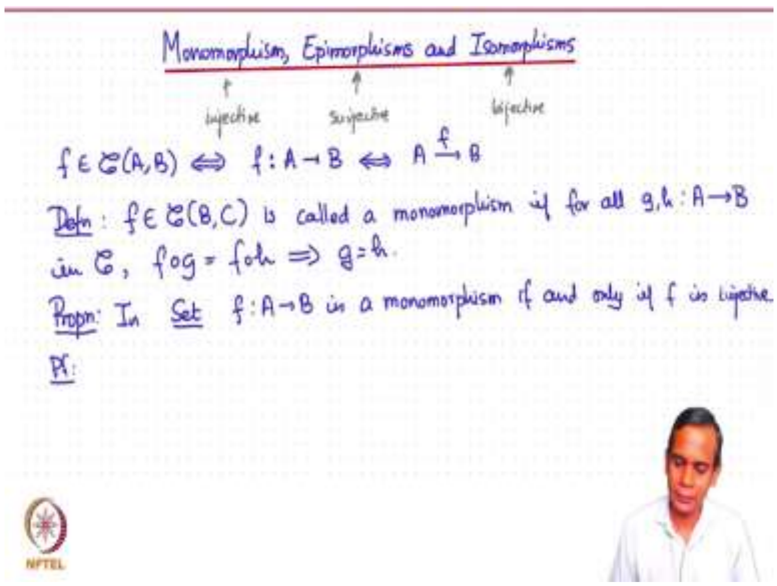

Conversely suppose f from B to C is not injective so what does that mean? It means that you can find points b1 and b2 in B, such that f of b1 is equal to f of b2. Now take A to be a singleton set, just one point a defined g of, so let us define g from A to B by setting g of a is equal to b1 and h from A to B by h of a is equal to b2.

Then you can see that f circle g of a is equal to f of g of a which is f of b1 but that is equal to f of b2 which is f circle h of a, but g if clearly not equally to h because g of A and h of A are not the same. Therefore, we have that f is not monic. So, we have completed the proof of this result that in the category of sets an arrow is monic if and only if it is injective. So, the definition of an epimorphism is in some sense dual to the definition of a monomorphism.

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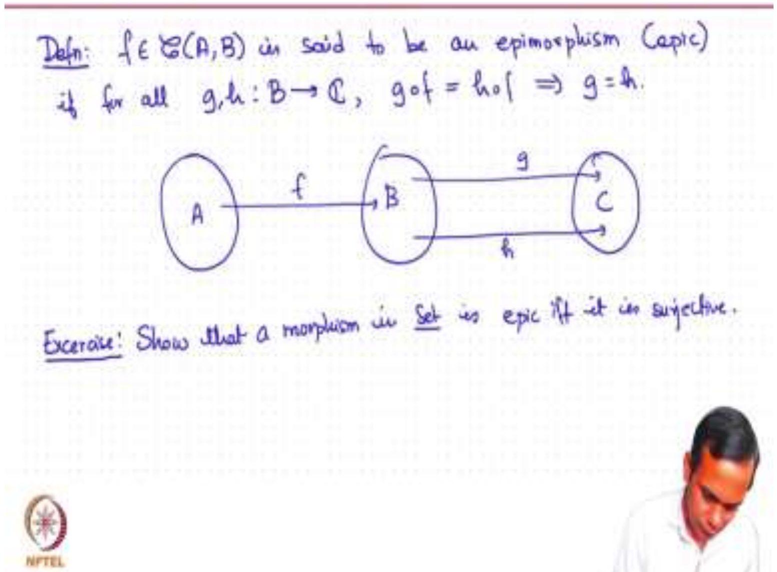

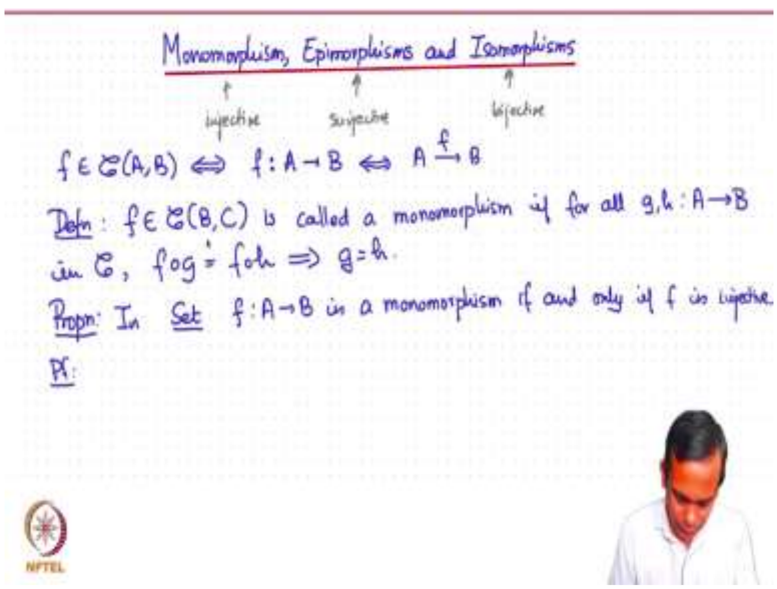

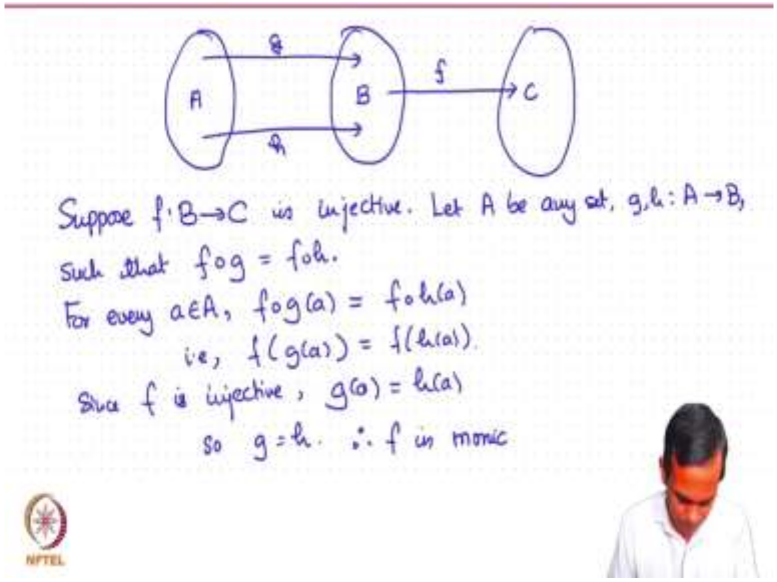

So, f an arrow in a category from an object A to an object B is said to be an epimorphism and the abbreviation for this is epic. If for all objects C and morphisms g comma h from B to C, if g circle f equals h circle f implies that g is equal to h. So, it is very similar to the definition of monic except that now the direction of the, the location of the arrow have changed.

So, let me draw picture analogous to this one. So, we have the, well so if we are looking at sets we could draw a picture like this, and we have f from A to B and we have g and h from B to C, and it is quite easy to show that in the category of sets a morphism is epic if and only if it is surjective. I will leave that as an exercise.

I would advise you all to actually sit and write this down. Show that a morphism in sets Set is epic if and only if it is surjective. But there two facts that monics are injective and epics are surjective this is now always true even in a category where the objects are sets and morphisms are functions, so we have seen a lot of categories where objects are sets and morphisms are functions. For example the category of groups, the category of rings, the category of modules for a fix strain and so on, but partly it is true.

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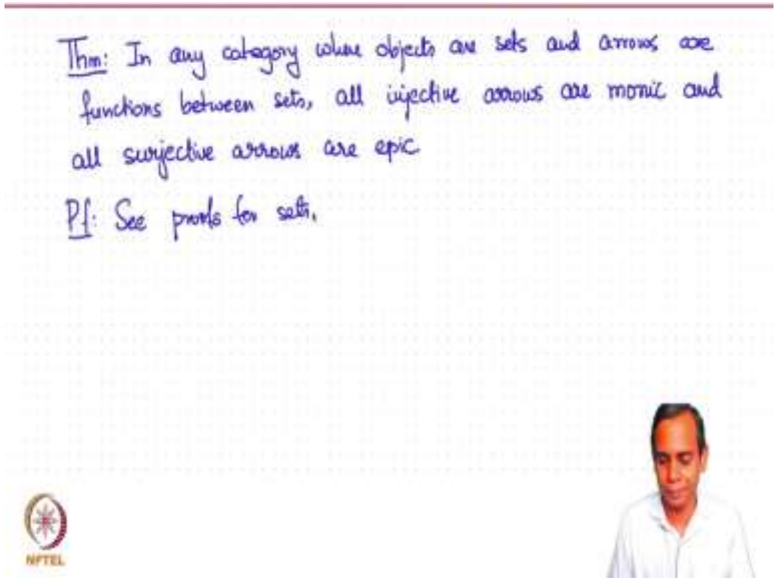

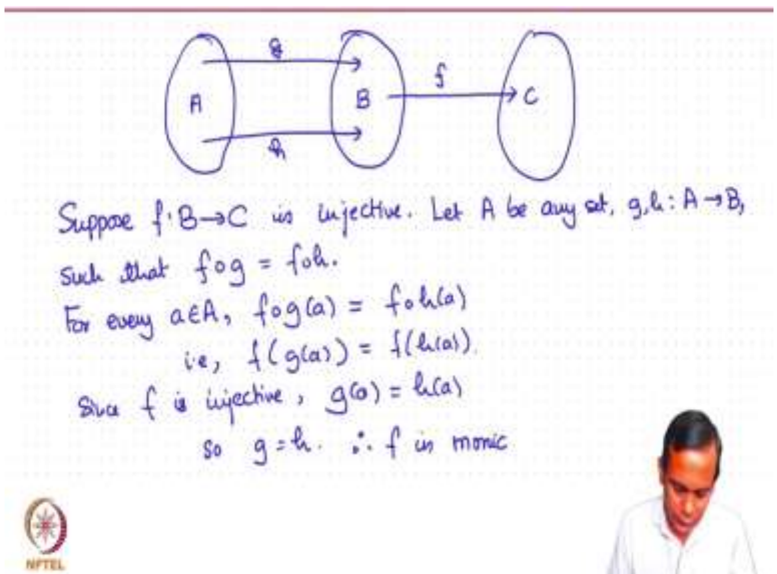

So, let us formulate this theorem. So, in any category where objects are sets and arrows are functions between sets, maybe with some additional conditions, for example the category of groups, the object each group has an underline set and the group homomorphism is a function from the set underline one group to the set underline another group.

Of course, it needs to satisfy certain more properties, it needs to preserve the identity multiplication and inverse and so on but it is such a category. Okay, all injective arrows or morphisms are monomorphisms are monic and all surjective arrows are epic.

Note that I have not claimed the converse it in fact turns out to be false. It is not true that every monomorphism in such a category is injective and it is not true that every epimorphism in such a category is surjective, but this part is fairly easy to prove.

So, actually I do not really need to prove this separately. It is, I have already proved it. So, if you look at this proof here, so here I am proving that in the category of sets every injective arrow is monic, but here I have not really used any property of the category of sets, this proof here holds equally well for any category where the objects are sets and arrows are functions.

If you just go through this proof again, suppose f is injective so f could be an injective group homomorphism but it is injective as a function, and we have g and h, they are arrows in the category, maybe in the category of groups or rings or whatever and, so they would be, satisfy some additional conditions, but they would still be functions, such that f circle g equals f circle h, so this whole argument would hold in, to prove the first part of this result.

That all injective arrows are monic. So, see proof for sets and for surjective to epic, well that is an exercise that I have given you. So, when you work out that exercise you should try to see that you are actually proving this theorem here. So, I will say see proofs for sets.

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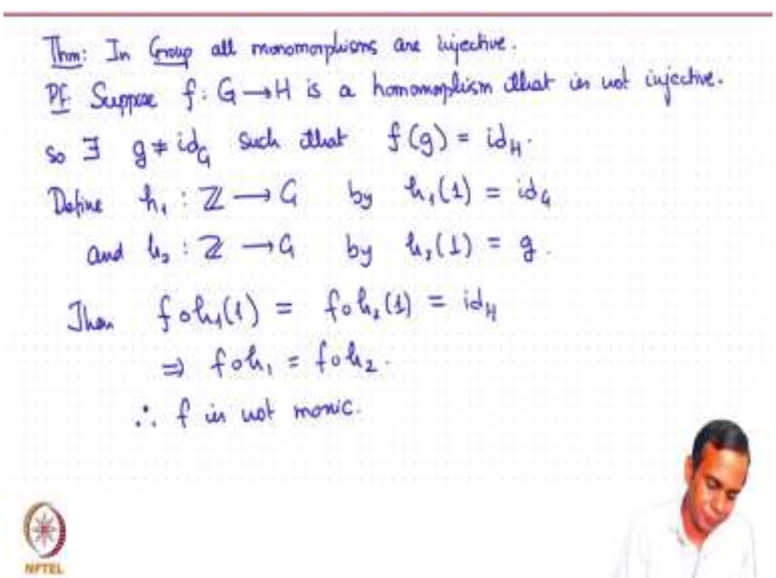

There are categories where all monomorphisms are injectives. So, here is an example of such a thing. Theorem in the category of groups which I denote group all monomorphisms are injective.

So, let us try to prove this. So, remember that the category group is the category whose objects are groups and whose arrows are group homomorphisms.

So, what we need to show is that if something is not injective then it is not monomorphism. Suppose that f from G to H is a homomorphism that is not injective, a group homomorphism that is not injective. So, now for such a homomorphism, so recall from the theory of groups that homomorphism is not injective if and only if its kernel is not trivial.

So, if it is not injective that means that its kernel is not trivial, so that means that there exists an element g which is different from the identity of g, such that f of g is equal to the identity of H. So, there is a nontrivial element in the kernel of f. Now, I will construct another group and two homomorphisms. So, define h1 and h2. I already unfortunately used up this element g here.

So, define h1 from Z to G and h2 from Z to G by h1 of, so a homomorphism from Z to G is completely determined by the image of 1. So, this is the group Z with addition as the operation by h1 of 1 is equal to identity of G and this one by h2 of 1 is equal to element g. Then f circle h1 of 1 is equal to f circle h2 of 1 and both of these are equal to identity of H.

And since every element in Z can be written as a sum of 1s or sum of the inverse of 1, this implies that f circle h1 is equal to f circle h2. So, what we have is that f circle h1 is equal to f circle h2 but h1 is not equal to h2. Therefore f is not monic. So, not injective implies not monic which is the same as saying monic implies injective. Therefore in the category of group all monomorphisms are injective.

So, monomorphisms are injective in the category of groups but there are counter examples, there are other categories where monomorphisms are not necessarily injective.

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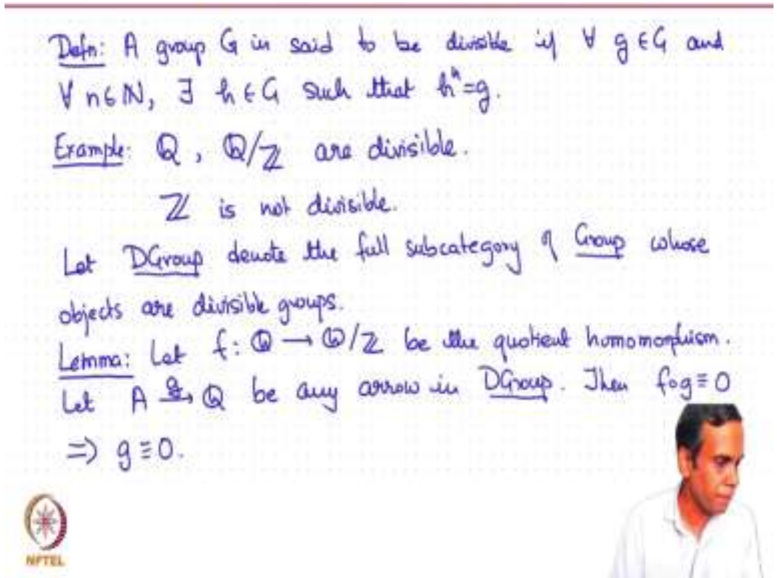

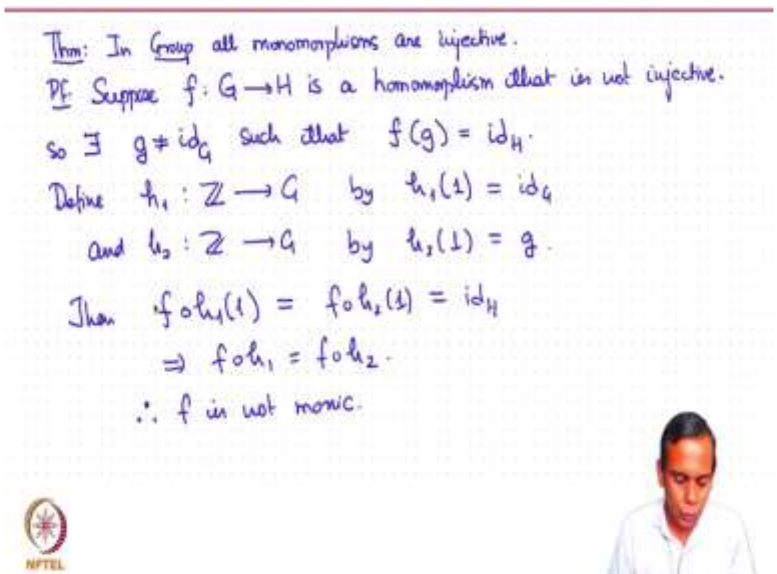

The simplest one, the simplest examples would be, let us just the category of divisible groups, so a group is said to be divisible if you can divide in it. So, if you can divide something in to equal parts. So, if for every element g in G so the G can be divided in to any equal parts.

And for every natural number N there exists an element h in G such that h to the n is equal to g. so for example Q is divisible. Any rational number can be divided by an integer M so the additive group of rational numbers is divisible and, so let us just, and another example is that quotient group of the rational numbers by the integers. This is also divisible.

What about groups that are not divisible, the group Z of integers is not divisible. Now let me define sub-category of the group, the category of groups. So, let DGroup denote the full sub-category of Group whose objects are divisible groups. So, this means that the objects of DGroup are only those groups which are divisible however the homomorphisms between any two objects in DGroup are all the group homomorphisms.

That is what it mean by full sub-category. Okay and now if we were to try to do this proof here that in group all monomorphisms are injective, this proof would fail in DGroup because here we have used the group Z which is not a divisible group. So, we would not be able to, this h1 and h2 could not be arrows in the category DGroup. They are only arrows in the category Group.

In fact it turns out that we can find examples here of monomorphisms that are not injective. So, here is a, so to do that let us just first prove a small lemma from which it will become clear that not all monomorphisms are injective. So, the candidate monomorphism that is not injective is the following map, f from Q to Q mod Z, just the quotient map, the quotient homomorphism.

Now let A to Q g be any arrow, so this f is an arrow in DGroup and let g be any arrow in DGroup. So, in particular that means that A is a divisible group and g is a group homomorphism. Then if f circle g is identically 0 then it implies that g is identically 0. So, if f circle g that will go from A to Q mod Z is identically 0, then g itself which goes from A to Q is identically 0. Let us prove this.

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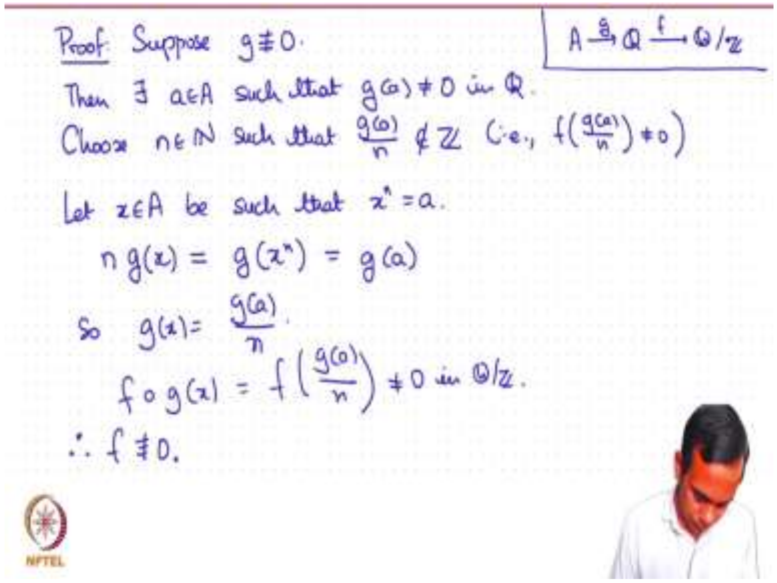

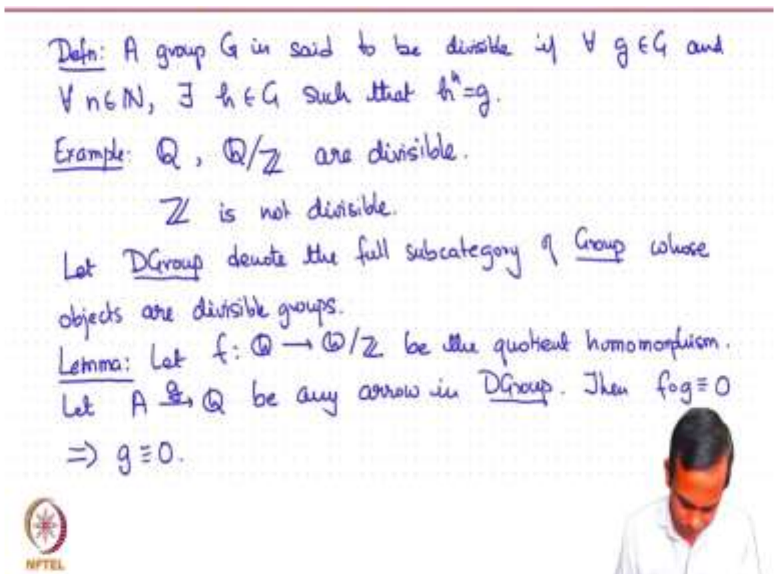

So, suppose we will again, we will prove the contra positive, suppose g is not identically 0, okay so recall that g is from A to Q. Let me just write down the name and things here, A to Q we have g and we have f from Q to Q mod Z. Okay so suppose g is not identically 0, then that means that there exists a in A such that g of a is a non 0 rational number.

Okay, now if you have any non 0 rational number it maybe an integer or it may not be an integer, but even if it is an integer you can divide it by some sufficiently large natural number so that it is no longer an integer. Choose n a natural number such that g a by n is not an integer, i e f of g a by n is non 0 in Q mod Z. Okay, so now we will use the fact that a is divisible.

Let x belongs to A be such that x to the power n is equal to a, because a is divisible we can do this and now let us look at g of x. So, if we look at g of x, n times g of x, now since g is a homomorphism this is the same as g of x to the power n, but that is the same as g of a and that is, so that implies that g of x is g of a divided by n.

So, what that means is that f circle g of x is f of g of a divided by n but f of g of a divide by n is not 0 in Q mod Z, and therefore what we have is f is not identically 0 because we found a point on which f is not 0. So, we have proved is that if you have, if f circle g is 0 then g must be identically 0. This I claim is enough to show that f is a monomorphism.

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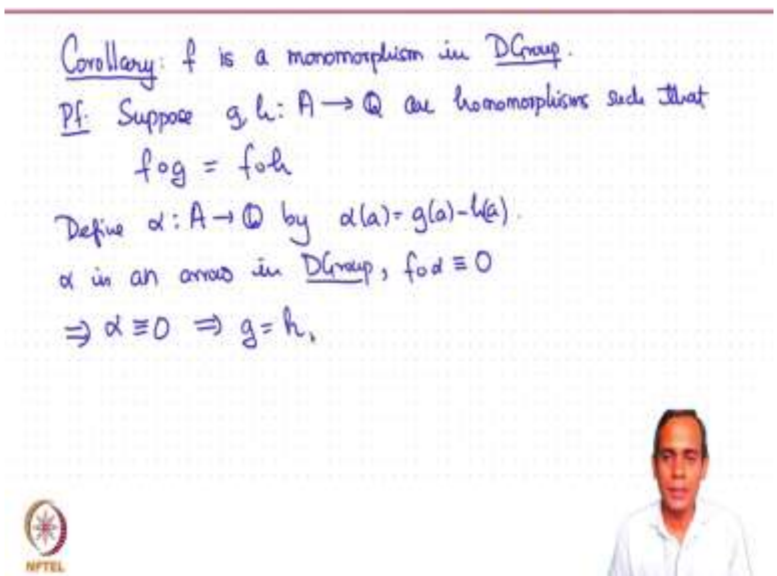

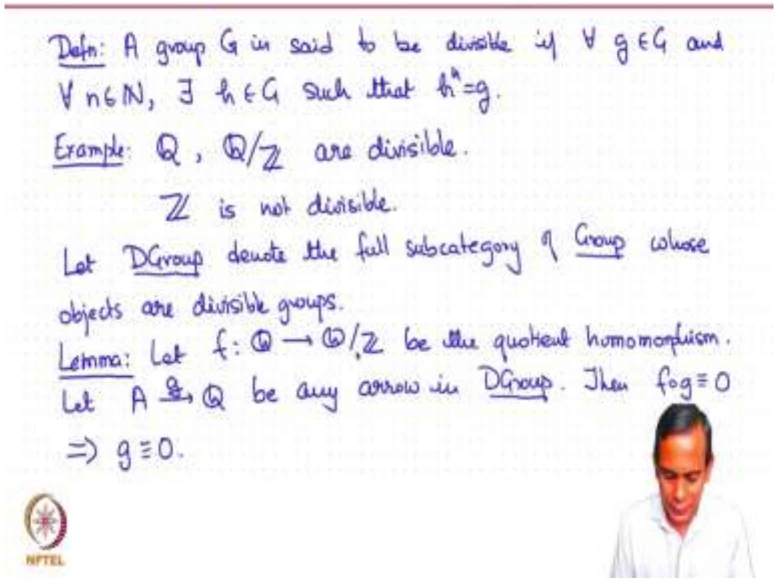

So, corollary f is a monomorphism in the category DGroup, category of divisible groups, and the proof, well so what we need to show is that if f circle g if f circle h then g is equal to h, so suppose g and h from some divisible group A to Q are group homomorphisms such that f circle g is equal to f circle h so that is a homomorphism A to Q mod Z.

Then define alpha from A to Q by alpha of a is equal to g of a minus h of a. So, this alpha is again a homomorphism, that is easy to check. So, it is an arrow in DGroup and what we have is because f circle g is equal to f circle A, f circle alpha is identically equal to 0 in, as a homomorphism from A to Q mod Z.

Yeah, but this implies by the previous lemma that alpha is identically 0 as a homomorphism from A to Q, but that is the same as saying that g is equal to h. So, what we have shown is that f is a monomorphism in the category of divisible groups. However, clearly you know f is, this quotient map it has a kernel it is not injective. So, what we see is that in the category of divisible groups not every monomorphism is injective.

So, what this underlines is that when you, see although in category theory monomorphism is supposed to be akin to injective maps and epimorphism and supposed to be akin to surjective map this is not exactly the case, and this underlines the importance of writing down proofs in purely category theoretic language for anything that you are going to state in general for categories.

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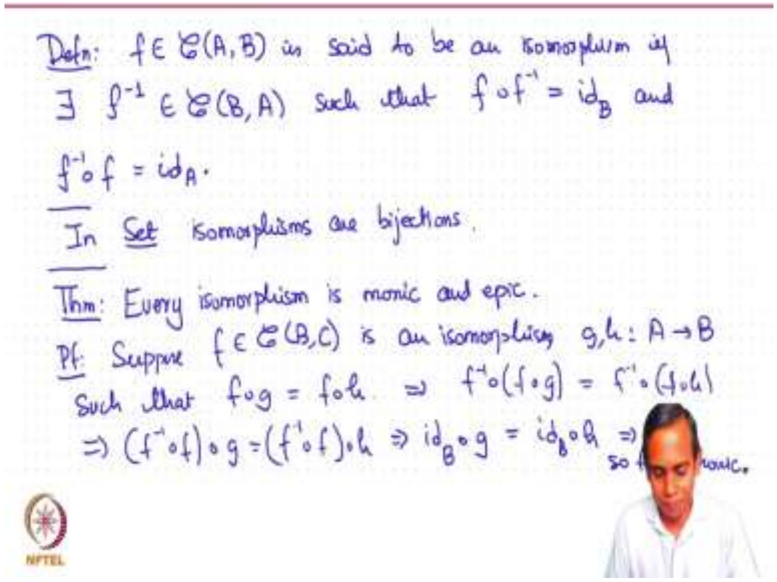

Finally we come to the notion of an isomorphism. So, f an arrow in a category C from A to B is said to be an isomorphism if, and this definition is much simpler than the definition of monomorphism and epimorphism. It just simply says that it has in inverse. So, if there exists an arrow f inverse from B to A such that f circle f inverse so f inverse is from B to A and f goes back from A to B.

This is actually the identity arrow of B and f inverse circle f is the identity arrow of A. So, this is a much more intuitive definition and clearly you know in the category of sets isomorphisms are precisely bijections, you have studied in set theory that invertible functions are one to one and on two. And so in the category of Set isomorphisms are bijections.

Now here is an easy result, theorem, every isomorphism in any category is monic and epic. The proof is very easy, let me do it for monic. Suppose f from A to B is an isomorphism, let us okay, so monic so let us say B to C. I like my objects to be in alphabetical order. And we have g and h from A to B such that f circle g is equal to f circle h.

What I can do is now I can precompose this f inverse circle f circle g is equal to f inverse circle f circle h. But by associativity this implies that f inverse circle f, circle g is equal to f inverse, circle f circle h. But by definition of f inverse this is identity of B, f inverse goes back to B yeah, identity of B circle g is equal to identity of B circle h but identity of B circle g by the identity axiom in category theory is g. So, this implies that g equals h so f is monic. I leave you to write down the proof that an isomorphism is epic as an exercise.