Algebra-II
Professor Amritanshu Prasad
Department of Mathematics
The Institute of Mathematical Sciences
Lecture 46
Definition of a category

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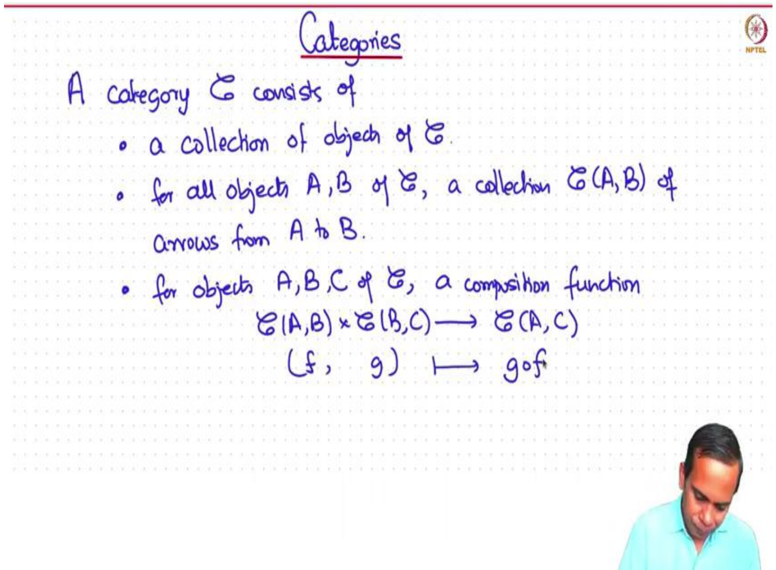

In this lecture I will explain to you the definition of a category and give you some examples. So, a category C consists of firstly a collection of objects of C and secondly for all objects A and B of C a collection C A, B of what we call arrows from A to B, and the third piece of data that we have given is for objects A, B, C of the category C and F, a function from C A, B cross C B, C to C A, C.

This function is actually a composition. So, we will call it a composition function, and this composition function is supposed to model the composition of functions. So, if f is a function from set A to the set B, and g is a function from set B to the set C, then g circle f would be a function from A to C.

So, that is the idea behind this, but here it is just an abstraction. So, you have given f here and g here you map it to what is the image is denoted by g circle f. So, a category has three pieces of data, a collection of objects for each pair of objects. A collection of arrows and whenever you have three objects when you have arrows from A to B and arrows from B to C, an arrow from A to B can be composed with an arrow from B to C, to give you an arrow from A to C. So, this is a three pieces of data and then there are certain axioms.

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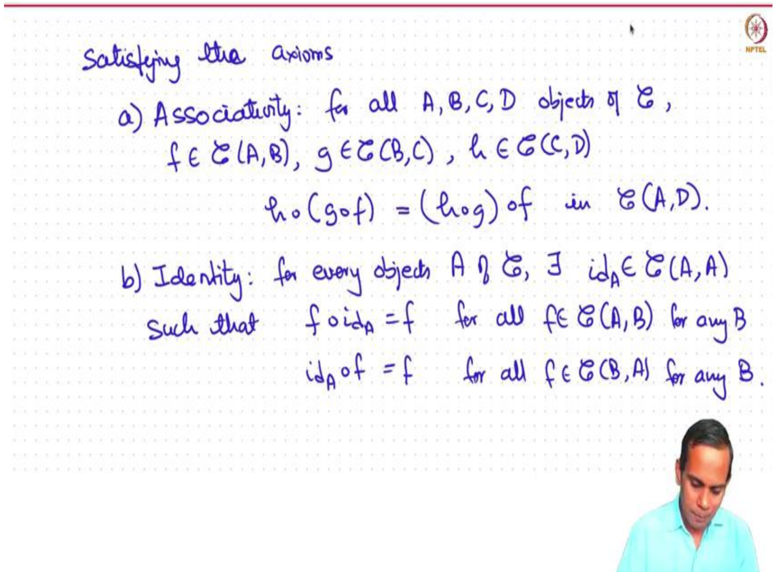

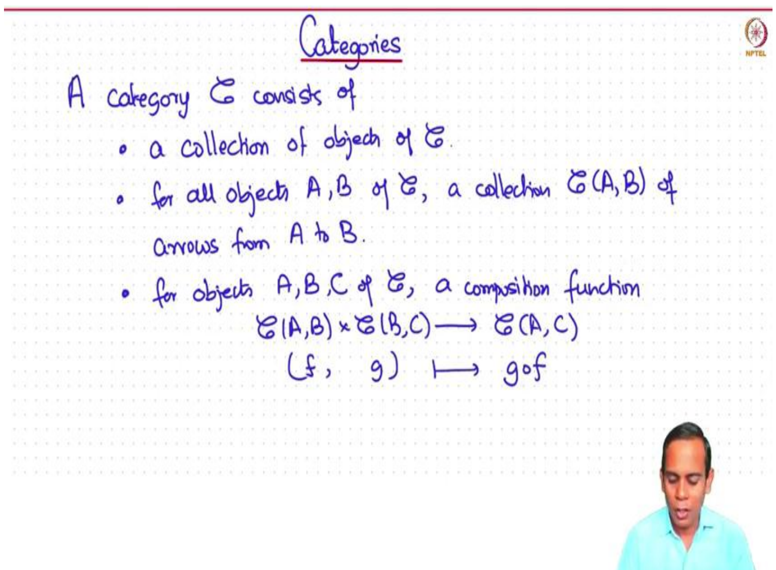

Satisfying the axioms. So, the first axiom is associativity for all objects A, B, C and D and f and arrow from A to B. g and arrow from B to C and h and arrow from C to D. you can compose these three into different ways. You can take h composed with g circle f or you could take h circle g and then circle f. So, this should be equal and there is an equality in C A, D and the second is the Identity Axiom.

It says that for every object A of C, there exists an arrow from A to A, such that if I take f circle identity of A this is equal to f for all f going from A to B for any object B of C, and identity of A circle f is equal to f for all f belonging to C B to A for any objects B, for any object B. These axioms now complete the definition of a category. So, category consists of objects and morphisms. Morphisms can be composed and they satisfy the associativity and identity conditions. Let us look at some examples.

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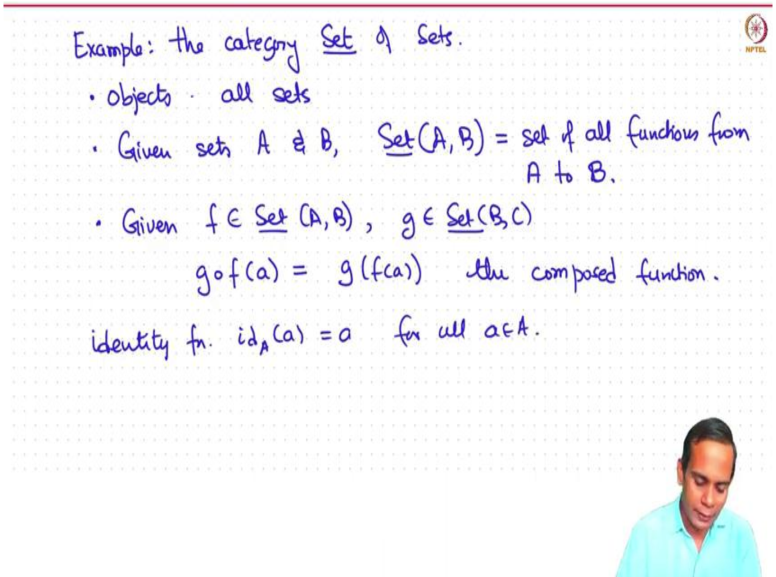

The most standard canonical example of a category is the category of sets. I will just denote by set with an underline. It is the category of sets. What are the objects? All sets. And this underlines why I said a collection of objects. I did not say a set of objects because when you start talking about the set of all sets you run in to paradoxes. So, objects are all sets and given sets A and B the arrows from A to B is the set of all functions from A to B. And what is the composition?

g circle f is the composed function. And we know from set theory that the composition of function is associative and of course every set has an identity function, which takes each element of that set to itself and this satisfies the with this identity function you can prove the identity axiom for categories. So, if you compose with the left or right, with the identity function you get the function that you are composing with. That is the most fundamental example. Let us look at more interesting examples that we have already encountered in our study of algebra.

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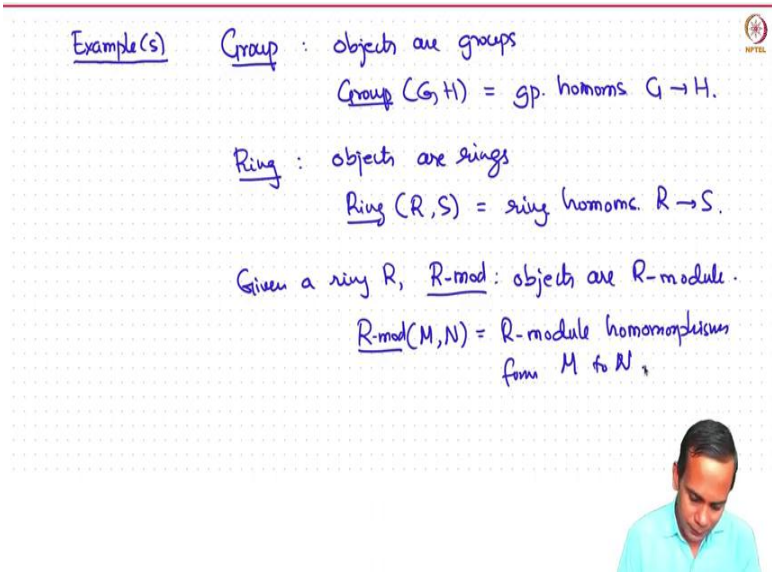

So, maybe this is examples you can look at group, objects are groups. Group G, H is equal to group homomorphisms from G to H, you can talk about ring, the category of rings and ring R, S equals ring homomorphisms from R to S. We can talk about given a ring R, we can talk about the category R-mod. Objects are R-modules and morphisms R-mod from M to N are R-module homomorphisms from M to N. In all these categories the arrows are actually functions. But the objects are all sets and the arrows are all functions. Let us look at a slightly different kind of example where the objects are not sets.

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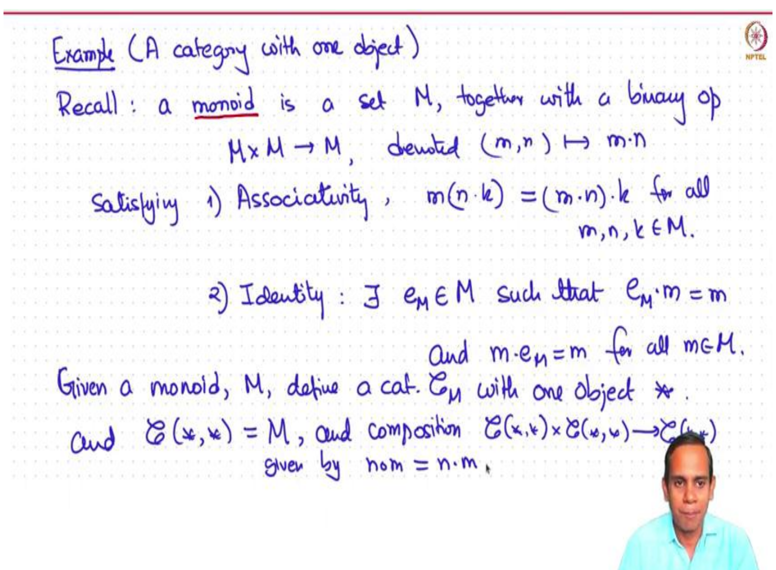

So, a category with one object. So, recall that a monoid, a monoid is an algebraic structure, it is a set M, together with the binary operation M cross M to M usually denoted m comma n goes to m dot n or just m times mn. Satisfying just two axioms, first is associativity which says that m nk, you can compute this in two ways, is equal to mn k for all m, n, k in M. And the second is the identity axiom, which says that there exists a distinguished element maybe I will call it, yeah, let us just call it eM belonging to M such that eM dot m is equal to m and m dot eM is equal to m, for all m in M.

So, these two axioms are sort of parallel to the axioms or a category and not surprisingly we can construct a category that is associated to this monoid. Given a monoid M define a category CM with just one object usually we denote it by star and the arrows from M to M is, from star to star is just the set of elements of the monoid M and the composition map is given by C star, star there is only object to worry about.

So, the only composition map that you have to worry about is from C star, star cross C star, star to C star, star given by n circle m is nm multiplication in the monoid. And then the associativity axiom for monoids corresponds to the associativity axiom for the category CM and the identity axiom for monoids, corresponds to the identity axiom for the category CM.

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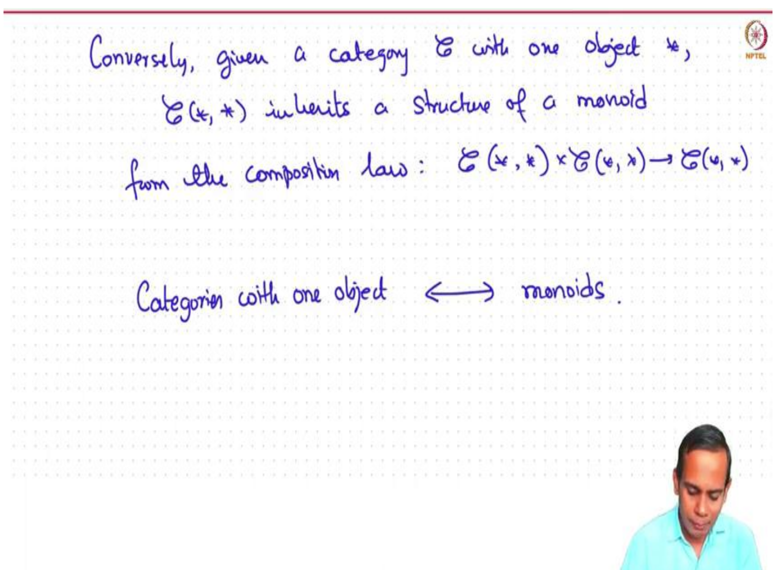

And conversely given a category with one object, let us call it star, C star, star, inherits the structure of a monoid by using the composition law. So, from the composition law, the category has a composition law and that give you a function from C star, star cross C star, star to C star, star and this is a binary operation which gives C star, star the structure of a monoid. It is a monoid because the associativity law for the category gives you the associativity law for the monoid and the identity axiom for the category gives you the identity axiom for the monoid. So, what we see is that categories with one object are the same as monoids. Now let us look at somewhat synthetic appearing example.

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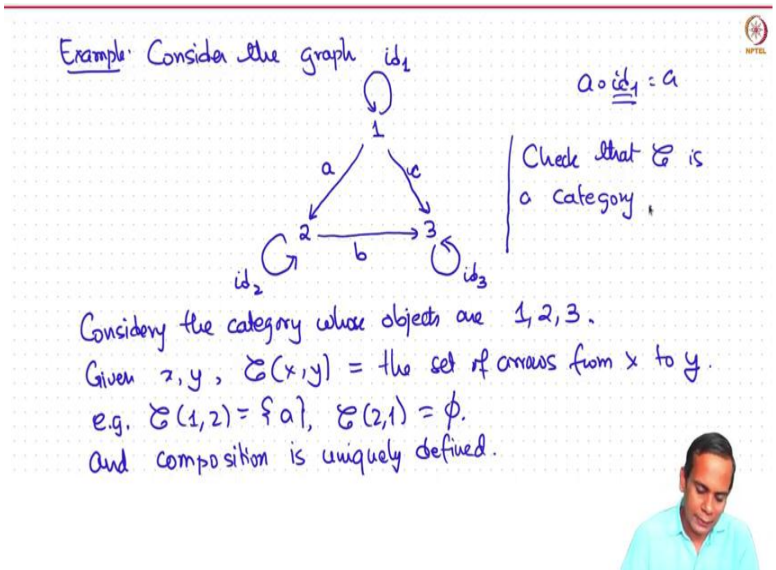

So, consider the directed graph, this sort of a visualization of a category. So, this graph has three vertices, 1, 2, 3 and we have an arrow from 1 to 2, an arrow from 2 to 3 and an arrow from 1 to 3 and we have an arrow from 1 to 1, we have an arrow from 2 to 2 and an arrow from 3 to 3. So, here I will call this arrow a, I will call this arrow b, I will call this arrow c. I will call this arrow identity 1, I will call this identity 2 and I will call this identity 3. You can probably guess the category that I am going to construct.

Consider the category whose objects are the three points 1, 2 and 3. In this case the objects are again not sets, they are just abstractions. I mean they are just symbols and given two objects C x comma y is the set of arrows from x to y. And so, for example C 1 comma 2 is just singleton a, whereas C 2 comma 1 well there are no arrows from 2 to 1, so this is the empty set. So, it is quite possible in a category for the collection of arrows between two objects to be empty.

And composition is sort of uniquely defined, because between any two objects there is only one arrow so if you want to compose b circle a then you must, it must be c because there is only one arrow from 1 to 3. There is only one possibility for composition. Now you need to check that composition is always defined that is given any two arrows in this category you can compose them.

So, for example, we know compatibly, so if I have an arrow from 1 to 1 and arrow from 1 to 2, I can compose them a circle identity 1 is a and so this in fact is the identity arrow of the object 1 and so on. So, I leave the details to you, check that C is a category. This is a special case of a more general category that is associated to a partially ordered set. When we studied Zorn's lemma, the definition of a partially ordered set we will use that to construct the category.

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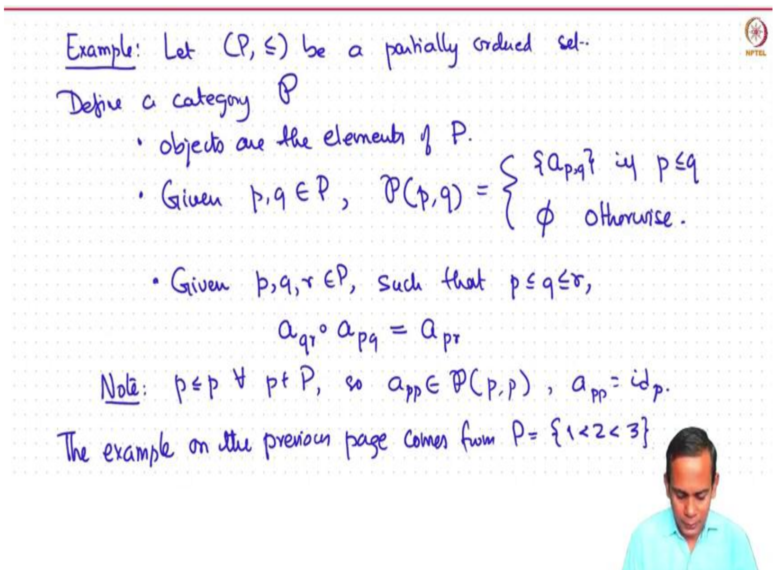

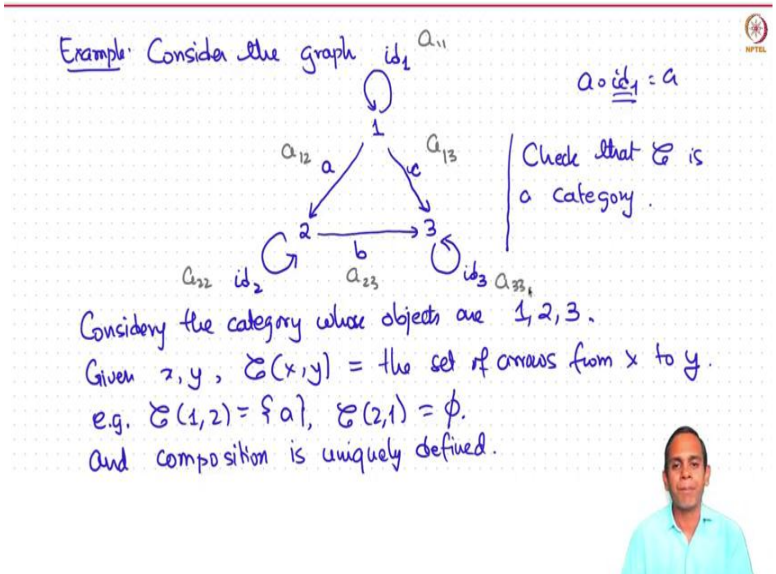

Let P be a partially ordered set. Define a category which we will denote by script P whose objects are the elements of P and given p, q in P are, I want to say in this coset P the arrows from p to q, there is going to be only one arrow which I will denote by say a p q if p is less than or equal to q and no arrows if otherwise. And the composition is given by, so now the only way that you can have an arrow from p to q and an arrow from q to r is that p is less than or equal to q and q is less than or equal to r.

Then the composition of a qr with a pq is the only possibility which is a pr. There is only one such possibility. And of course note that in a coset p is less than or equal to p for every p and coset P. So, there is a pp in P pp and a pp is the identity of p. With this you can easily check the axioms for a category are satisfied. Now this category that I defined on the previous example, this is a special case of this category associated to the coset.

The example on the previous page comes from the set which has three elements and 1 is less than or equal to 2 is less than or equal to 3. It is a total order on 1, 2, 3 and of course 1 is also, less than or equal to 3. So, that relation gives rise, of course 1 is less than or equal to 1, 2 is less than or equal to 2 and 3 is less than equal to 3. So, maybe the best way to write is like 1 is strictly less than 2 is strictly less than 3, that is the linear order and if you look at it, this is what we would call a12 this what we would call a13 and this is a23, this is a22, this is a11 and this is a33.

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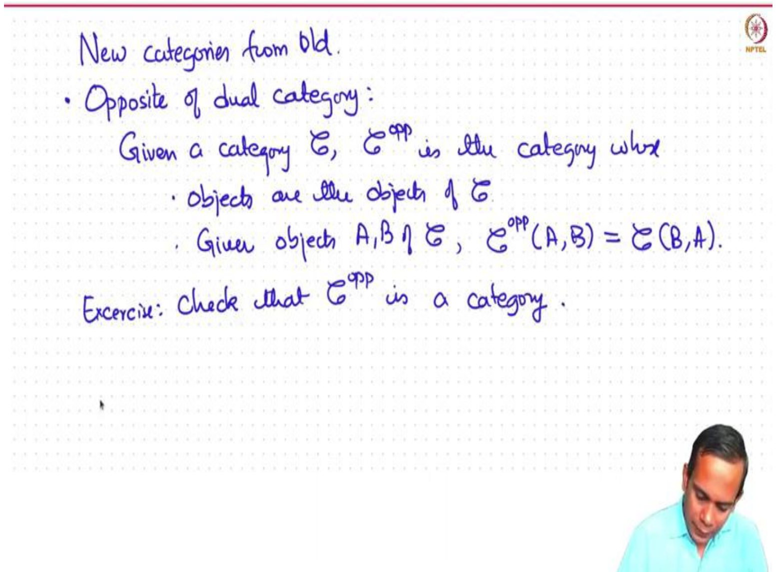

The next few examples are really ways of constructing new categories from old ones. So, the first is the opposite or dual category. So, given a category C, C opp is the category whose objects are the objects of C. And given objects A and B, which are also objects of C opp the arrows from A to B in C opp are declared to be the arrows from B to A in C. And the associativity law for C turns in to the associativity law for C opp and identity axiom also turns in to the identity axiom for C opp. So, I will leave it as an exercise for you to check that C opp is also a category.

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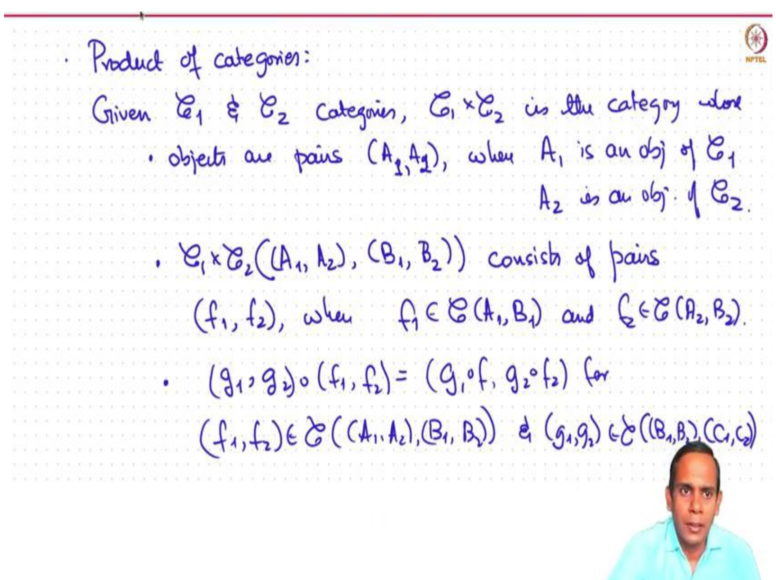

The next construction of a new category from the old is the product of categories. So, given two categories C1 cross C2 is the category whose objects are pairs A1 comma A2 where A1 is an object of C1 and A2 is an object of C2. And given, so we should also say what morphisms are, and so C1 cross C2 the morphisms from, this is A1, this is A2.

So we have an object A1 comma A2 and another object B1 comma B2 then a morphism from A1 comma A2 to B1 comma B2 is a pair consists of pairs f1 comma f2 where f1 is an arrow from A1 to B1 and f2 is an arrow from A2 to B2 and the composition law is given by g1 circle g2, no g1 comma g2 circle f1 comma f2 is g1 circle f1 comma g2 circle f2 for f1, f2 in C A1, A2 to B1, B2 and g1, g2 in C B1, B2 to C1, C2 for any objects A1, A2 of, A1, B1 and C1 of the category C1 and A2, B2, C2 objects of the category C2.

And again it is fairly straightforward to check that with this specification we actually get a category. So, the associativity and identity axioms are inherited from the categories C1 and C2. Now we come to the definition of sub categories, which is the last construction of a new category from an old one.

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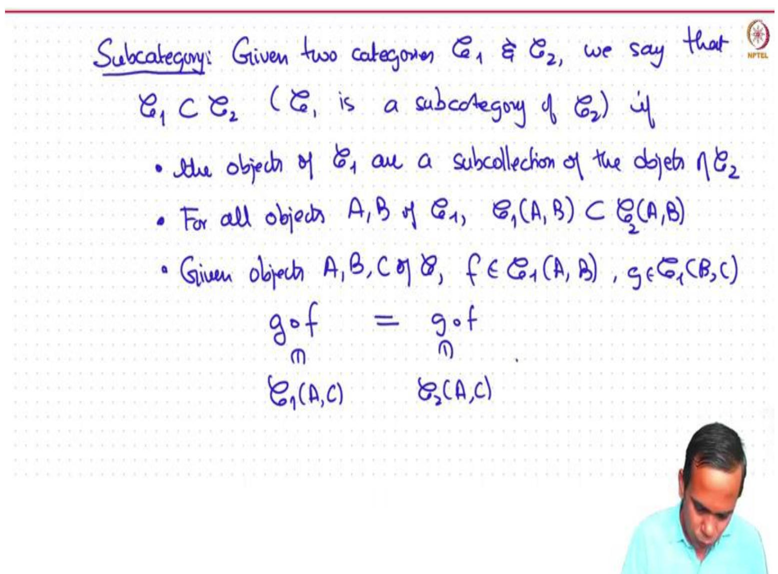

Given two categories C1 and C2 we say that C1 is a sub category of C2 and we just use the sub set notation for this, if, well the objects of C1 are a sub-collection of the objects of C2. Then for all objects A and B of C1, the arrows in C1 from A to B is a sub-collection of the arrows in C2 of A to B, and the third property is that given objects A, B, C of C1 and morphisms f C1, A, B, g

C1 B, C we have g circle f. This is the composition in C1 A, C must coincide with g circle f in C2 A, C. In other words, the composition maps are compatible. This is an example of a sub-category. Let us look at some concrete examples of sub-categories.

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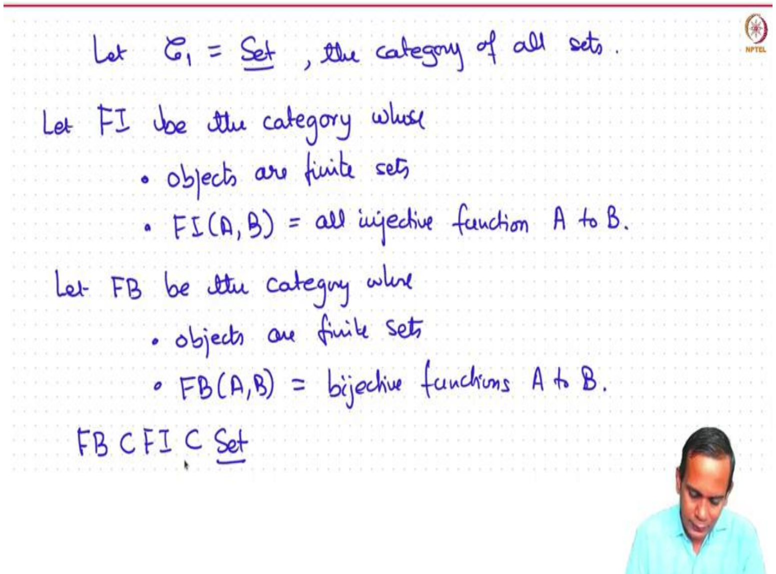

So, the first one is, so let C1 be the category of all sets, and I will define a sub category of it. FI, FI is short for Finite Injective, be the category whose objects are finite sets and arrows FI A, B is all injective functions from A to B. And let us define another category, FB be the category whose objects are again finite sets and FB A, B are bijective functions A to B. We need to check that FI and FB are indeed categories, but that is not difficult. You just need to know that a composition of injective function is injective and of course the identity function is injective and here the composition of bijective function is bijective and identity function is bijective.

And we have a hierarchy of categories here. FB is a sub-category of FI and FI is a sub-categories of the category of sets and there is a slight difference between these containments here, even the objects are not the same but we are restricting ourselves to finite subsets. So, the collection of objects here is smaller than the collection of objects here but in this containment the collection of objects here is the same as the collection of objects here.

What we have changed are the arrows. So, in the definition of a category it is not just important what the objects are, it is equally important what the arrows are. And now let me give you one last thing which I can put in between here and I will define a category of finite sets.

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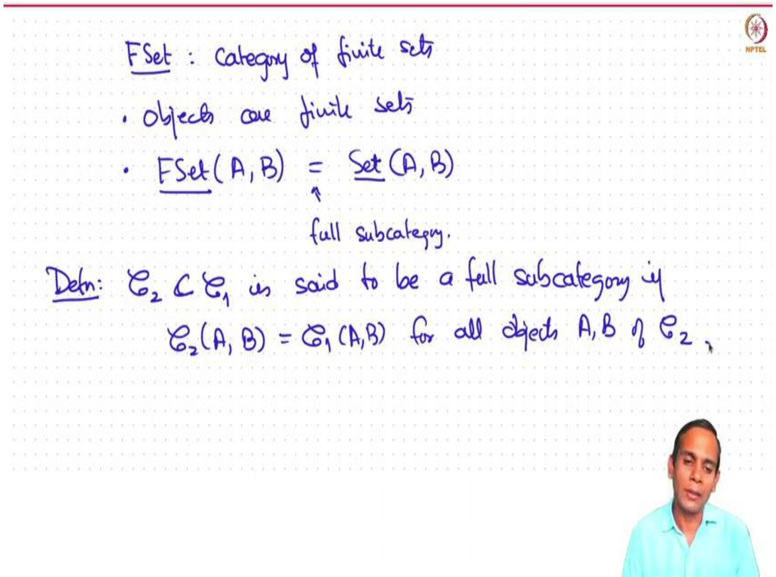

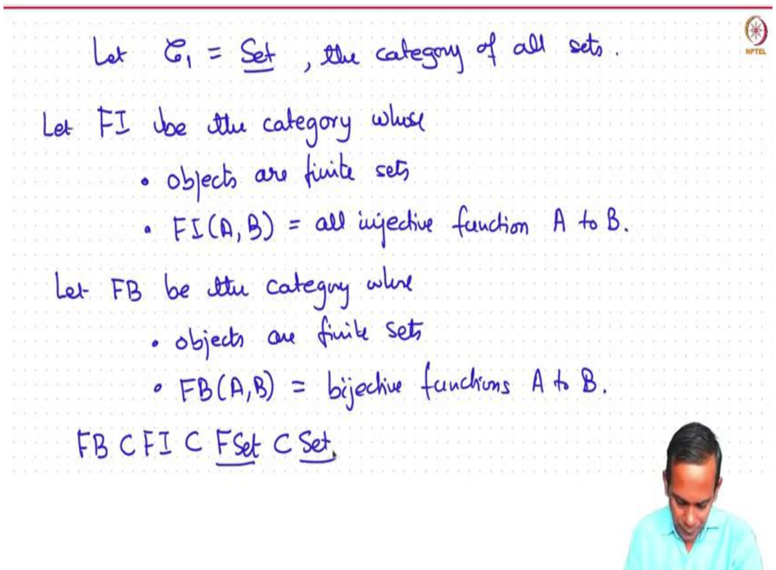

So, F Set is the category of finite sets, objects are finite sets and if you have two finite sets the arrows from A to B are the same as the arrows from A to B in Set of in the category of sets to all functions from A to B. When you have this condition, what you, this is the notion of a full sub-category. So, you have only restricted the objects but whenever you have two objects in your sub-category the arrows are the same as in the larger category.

What we have is that F Set is, sits in between FI and Set. And so we can formalize this notion of full sub-category into a definition. C2 a sub-category of C1, is said to be a full sub-category if C2 A, B is C1 A, B for all objects, A, B of C2. So, the objects get restricted but the arrows do not.