Algebra – II
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Lecture 3
Isomorphic Extensions

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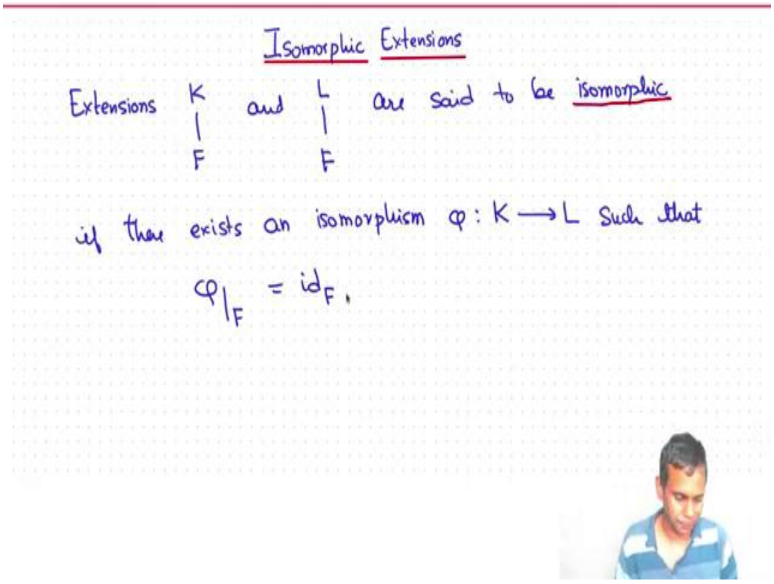

Extensions, say K over F and L over F are said to be isomorphic, if there exists an isomorphism offerings. But there is an additional condition that phi restricted to F is equal to the identity function from F to F. So, for all practical purposes two field extensions are thought of as the same if you can find an isomorphism of the larger fields, which keeps the elements, which maps each element of the smaller field into the corresponding element of the smaller field in the other extension. Let us look at an example.

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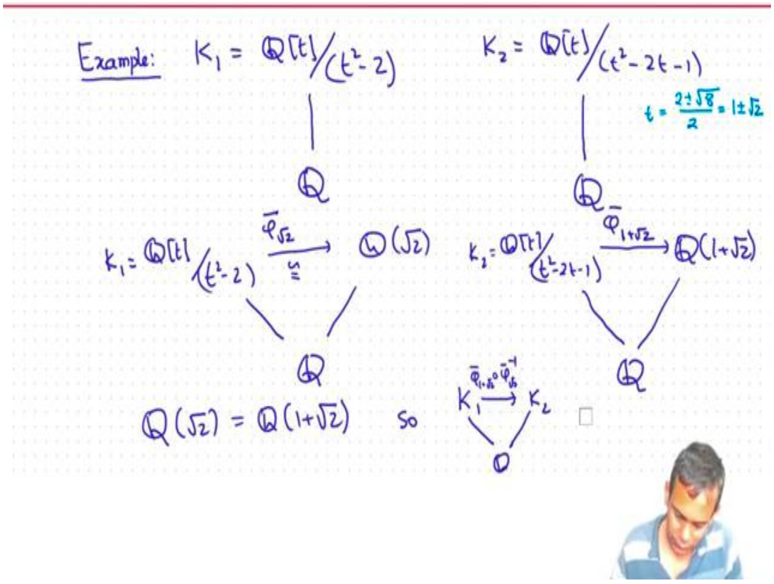

Look, let us look at K1 equals Q t mod t squared minus 2. This, we are familiar with t squared minus 2 is an irreducible polynomial. And so, this ideal t squared minus 2 is a maximal ideal in Q t. And so, Q t mod t squared minus 2 is a field and it contains Q. And Let us construct another 1, K2. And let us say this is Q t mod different polynomial t squared minus 2 t minus 1. So, this again is an extension over Q. And let us see if we can figure out whether these two extensions are isomorphic or not. Now, on the face of it, they both have you know, if you think of these as vector spaces of a Q, they both have degree 2, so they could be isomorphic.

Now, we already have some understanding of this first field extension, we know that, if you take Q t mod t squared minus 2, this is isomorphic to Q root 2 over Q, why? Well, you just take the map phi bar of root 2. We saw that this gives rise to an isomorphism. And the way we define it, the elements of Q go to elements of Q because they just correspond to constant polynomials on this side. So, this is an isomorphism. So, this is K1. And what about K2, so let us just see what the roots of this polynomial t squared minus 2 t minus 1 are.

Well, if you just think about it, the roots are t equals 2 plus or minus square root 4 plus 4, 8 by 2, which is 1 plus or minus root 2. So, in a similar manner, we can think of this K2 as Q t mod t squared minus 2 t minus 1 with using the map phi bar, say 1 plus root 2. This goes to Q 1 plus root 2. And this is an isomorphism over Q. But of course, Q root 2 is the same as Q 1 plus root 2. Why is that?

Well, if you have a field that contains root 2, and it contains Q, then it must contain 1 plus root 2. Conversely, if you have a field that contains 1 plus root 2 and contains Q, then it must contain root 2. So, these are actually the same, every field that contains Q and root 2 is also a field that contains Q and 1 plus root 2 and vice versa. And so, we have that K1 and K2 are isomorphic over Q. We have this isomorphism, which we can call, let us say phi bar 1 plus root 2 circle phi root 2 bar inverse is an isomorphism from K1 to K2, and these are isomorphic over Q.

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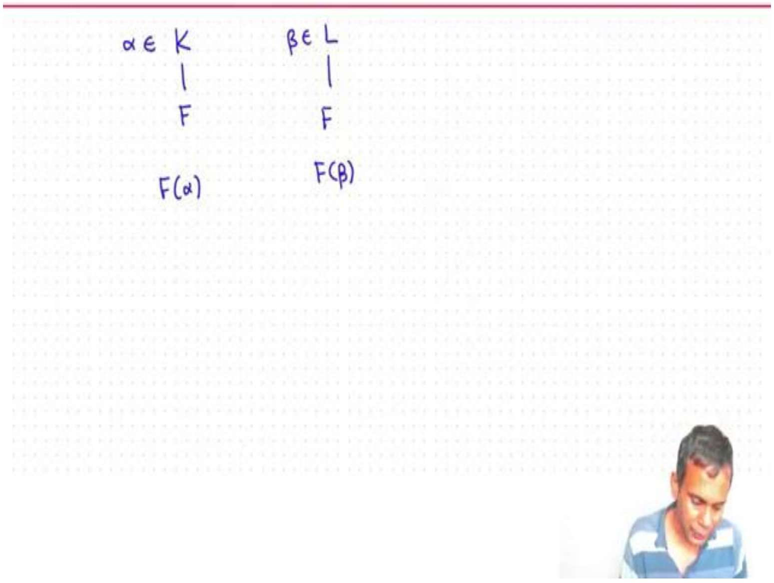

Now, suppose we have 2 field extensions, let us say, K over F. And we have an element alpha here, and another field extension L over F an element beta over here. So, then we can talk about the field F alpha, we can talk about the field F beta, which lies over there. So then, in general, is not very easy to figure out whether or not F alpha and F beta are isomorphic over here. But if you look at a slightly more restricted notion of isomorphism, then there is a very beautiful and simple answer.

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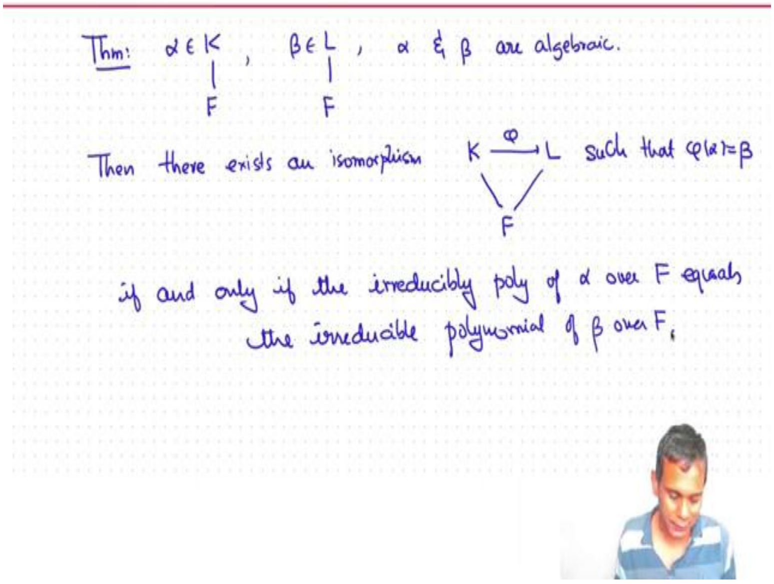

So that is the next theorem. So, now, suppose we have this situation, where we have an alpha in a field extension over F, beta in a field extension L over F, and alpha and beta are algebraic. See F alpha and beta are both transcendental then, of course, F alpha is isomorphic to the field of rational functions over F, F of beta is also isomorphic to that same field and so F alpha is isomorphic to F beta. And if alpha is algebraic but beta is transcendental then F adjoint alpha is a finite-dimensional vector space over F, whereas F beta, when beta is transcendental is going to be infinite dimension.

So, we know that F alpha cannot be isomorphic to F beta. So the only interesting case for the isomorphism question is when alpha and beta are algebraic. So, case of the statement is that then there exists an isomorphism phi from K to L. So, I am going to draw it like this. So, this means that phi is an isomorphism from K to L, but when restricted to F phi is the identity, so it is an isomorphism of extensions, not just a ring isomorphism. But we are going to add an additional condition such that phi of alpha is equal to beta.

Now, once you add this condition, this decision problem becomes much easier, then, if and only if the irreducible polynomial of alpha over F equals the irreducible polynomial of beta over F. So, when you put this additional condition that phi must map alpha to beta, then this problem becomes much simpler. And see the reason why this particular problem is more interesting, is very interesting when we study Galva theory.

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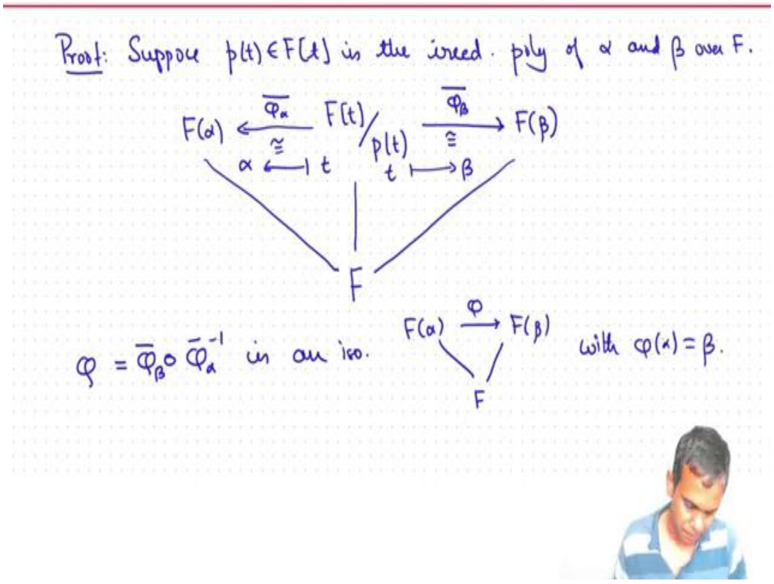

So, here is the proof. So, to start with, let us assume that alpha and beta have the same irreducible polynomial over F. So, suppose p t, belongs to F t is the irreducible polynomial of alpha as well as beta over F. So, then, we have F t mod p t, on the one hand, this phi alpha bar gives an isomorphism to F alpha phi beta bar gives an isomorphism to F beta. And these are all isomorphisms over F. And under this isomorphism, t goes to alpha under this isomorphism t goes to beta. And so now if you take phi beta bar circle phi alpha bar inverse then this is an isomorphism from F alpha to F beta over F with phi of alpha, where alpha goes to t, t goes to beta equal to beta that proves one way.

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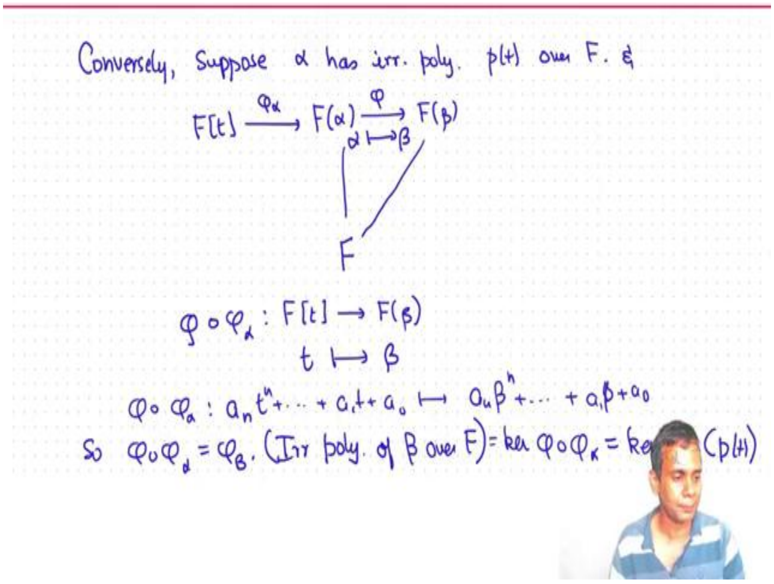

And for the converse suppose that alpha has irreducible polynomial p t over F. And suppose that, we have an isomorphism from F alpha to F beta, is an isomorphism, which takes alpha to beta. Now, we have from F t, we have the map phi alpha, the substitution map to F alpha. And if you consider the map phi circle phi alpha, this is a homomorphism from F t to F beta. And this takes t to beta. Since it is a ring homomorphism, it also takes a n, t to the n plus a1 t plus a0 to a n beta to the n plus a1 beta plus a0.

So, this gives rise to map from F t to F beta, but you recognize this as the map where you are substituting for each polynomial the variable t to be beta. So, what we have is that phi circle phi alpha is just phi beta. So, the irreducible polynomial of beta over F is the kernel of phi circle phi alpha, but that is the kernel of phi alpha because phi is an isomorphism. But this is the generator of the kernel of the ideal generated by the irreducible (()) (13:20) beta over F is this but this is the ideal generated by p t and so, p t must be the irreducible polynomial of beta over F.

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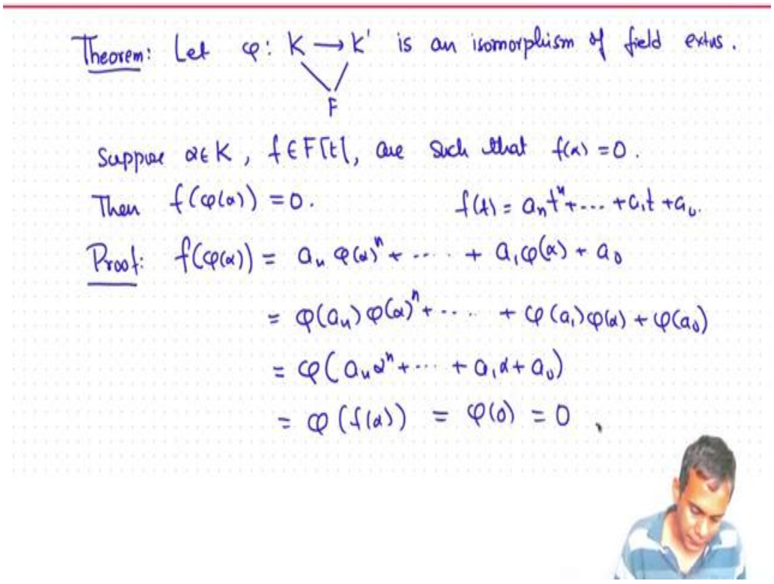

An important property of isomorphisms of extensions is that if you take an element in one extension with a certain irreducible polynomial and you map it under an isomorphism to another field extension, then its image will have the same irreducible polynomial in the second extension. Let me state this precisely suppose, phi from K to K prime over F is an isomorphism of field extensions. And suppose alpha belongs to K is an element such that an f is a polynomial in F t are such that f of alpha is 0 then f of phi alpha is also going to be 0. This is rather straightforward. So, let us just look at f of phi alpha.

So, let us say that f of t is the polynomial a n t to the n plus dot dot dot plus a1 t plus a0, then f of phi alpha is a n phi alpha to the n plus dot dot dot plus a1 phi of alpha plus a0. Now, a n, a n minus 1 up to a0, these are all elements of F. So, phi of ai is equal to ai. So, I can write this as phi of a n phi of alpha to the power n plus dot dot dot plus phi of a1 phi of alpha plus phi of a0. Now we are applying phi to all the terms here and they are just sums and products. So, because phi is a ring homomorphism, this is the same as phi of a n alpha to the n plus a1 alpha plus a0, which is just phi of f of alpha, with that is phi of 0, which is 0. That is the proof.

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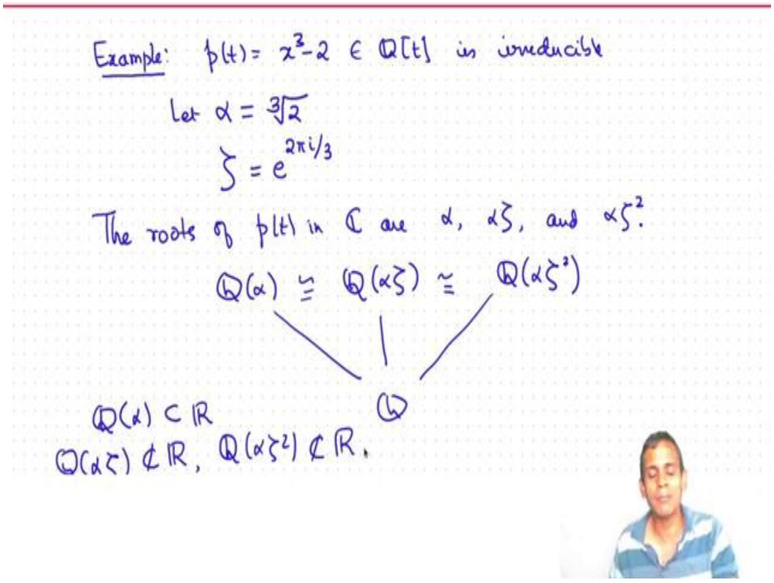

Now let us look at an example. So, consider the polynomial p t equals x cubed minus 2. So, this is an irreducible polynomial in Q t. This polynomial has no rational roots, what are the roots of this polynomial? So, let us say alpha equals cube root of 2. And Let us say zeta is equal to e to the 2 pi i over 3. So, this is a cube root of 1 in the complex numbers, zeta cube is equal to 1. Then the roots of p t are in the complex numbers, alpha, alpha zeta and alpha zeta square. So, we have 3 fields, we have Q alpha, Q alpha zeta and Q alpha zeta squared.

And each of these elements alpha, alpha zeta, and alpha zeta squared, they all have the same irreducible polynomial over Q, namely, the polynomial x cubed over 2, x cube minus 2, x cube minus 2 is the irreducible polynomial of all these 3 elements over Q. And therefore, these are all isomorphic over Q. These 3 extensions are isomorphic over Q. But notice that Q alpha is a subfield of the real numbers, whereas Q alpha zeta is not a subfield of the real numbers and Q alpha zeta squared also is not a subfield of the real numbers.

And so, this means that if you want to think about these field extensions properly, it is important to sort of free them from the context of the real numbers and think of them as sort of abstract objects. And this is where the notion of isomorphism of extensions comes into being. So, an abstract property of a field extension something that is invariant under field isomorphism being a subfield of the real numbers is not such a property.