Algebra II
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Uniqueness of Algebraic Closure

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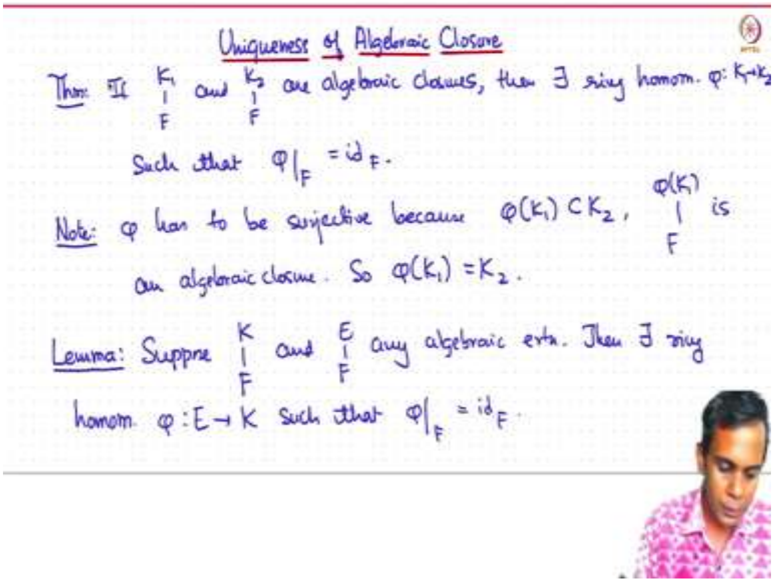

We are now ready to prove the Uniqueness of Algebraic Closure. The statement has to be made somewhat carefully. It says that if K1 over F and K2 over F are two algebraic closures of F, then they are isomorphic over F, which means that there exists a ring homomorphism phi from K1 to K2, such that phi restricted to F is the identity function on F.

Note that the existence of a ring homomorphism will automatically mean with, with satisfying this condition will automatically mean that this ring homomorphism phi is an isomorphism. Firstly, such phi is going to be injective because its kernel is going to be an ideal of K1. And it is going to be a proper ideal of K1, because phi is not identically 0.

But the only proper ideal of K1 is singleton 0, and so phi has to be injective. And phi has to be surjective, that is a little more subtle, phi has to be surjective, just because if you look at phi of K1, this is a subfield of K2 and phi of K1 being an isomorphic image of K1 is an algebraically closed field. So, phi of K1 inside over F is an algebraic closure.

So, if you take any element of K2 it must lie inside phi of K1 because it is the root of a polynomial in F. So, we must have phi of K1 is equal to K2. So, all we need to do is show that we can construct such a ring homomorphism. Now what we will do is, we will prove a slightly more general lemma which is useful in its own right.

So, this lemma says that suppose we have an algebraic closure K over F and we have E over F any algebraic extension, not necessarily an algebraic closure, then there exists a ring homomorphism phi from E to K such that phi restricted to F is the identity function on F. So, this is a special, the theorem is a special case of this lemma where we take E to be K1 and K to be K2. The proof of this lemma proceeds by using Zorn's lemma.

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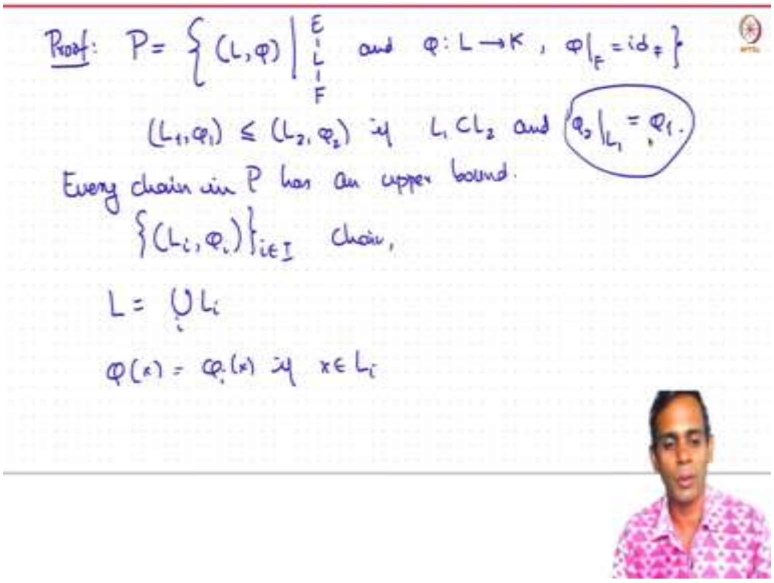

So, we must construct a partially ordered set to which we will apply Zorn's Lemma. I defined the partially ordered set to consist of all pairs L comma phi, where L is a field that lies between L and F and phi is a ring homomorphism from L to the algebraic closure, K2 L2 K such that phi restricted to F is equal to the identity element of F.

So, we look at this collection and we define a partial order say that L1 phi1 is less than or equal to L2 phi2, if L1 is a subset of L2, and if we take phi2 and restrict it to L1 then we get phi1. So, that is your definition of P. And it is easy to see that every chain in P has an upper bound. You just take if you have a chain Li phi i, just define L to be union Li and define phi of x to be phi i of x, if x belongs to Li.

And this phi is going to be well defined, it will not depend on which i I choose such that x belongs to Li. Because of this condition here that if I take two different such extensions, they are going to be comparable and the restriction of phi from the larger one to the smaller one is going to be the phi corresponding to the smaller one.

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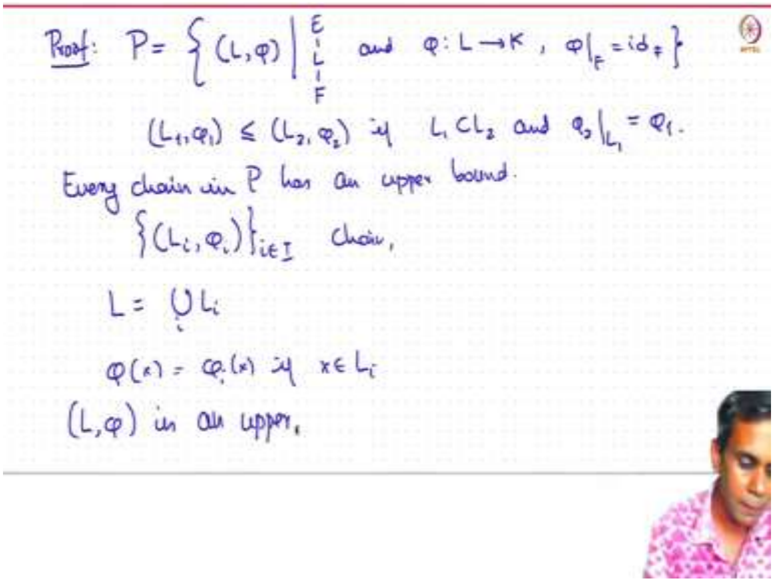

So, then L comma phi is an upper bound.

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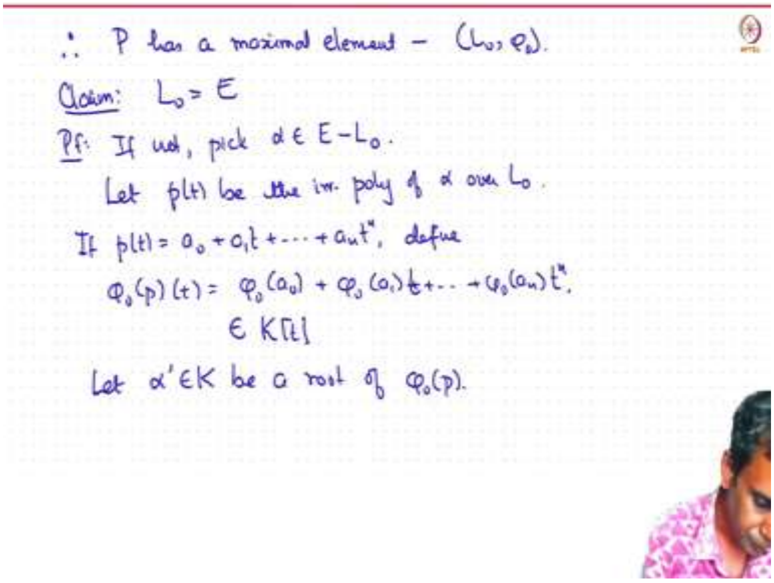

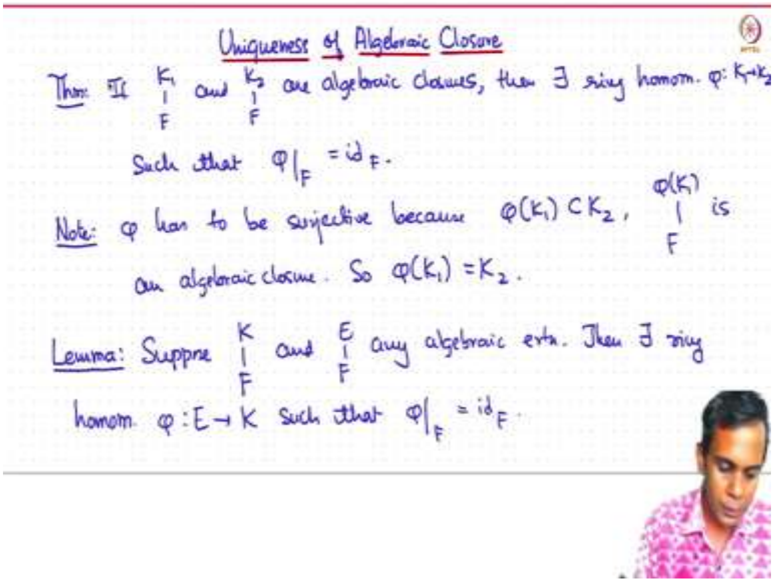

It follows by Zorn's lemma that P has a maximal element. Let us call it L0 comma phi0. I claim that L0 has to be equal to E. If not just pick any alpha which is in E, but not in L0. Now alpha is in E and E is algebraic over F. Therefore, alpha is algebraic over F, which means that alpha is also algebraic over L0. So, let Pt be the irreducible polynomial of alpha over L0. And now, I will define a map.

So, let us so we have now this phi0 from L0 into the algebraically closed field K. So, let it phi if Pt is the polynomial a0 plus a1t plus an t to the n define phi0 P of t to be the polynomial phi 0 a0 plus phi0 a1t, and so on plus phi0 an t to the power n. Now, this is a polynomial in Kt in an algebraically closed field, and therefore it has a root in K in K. So, let alpha be a root, no, let us call it let alpha prime in K be root of phi0.

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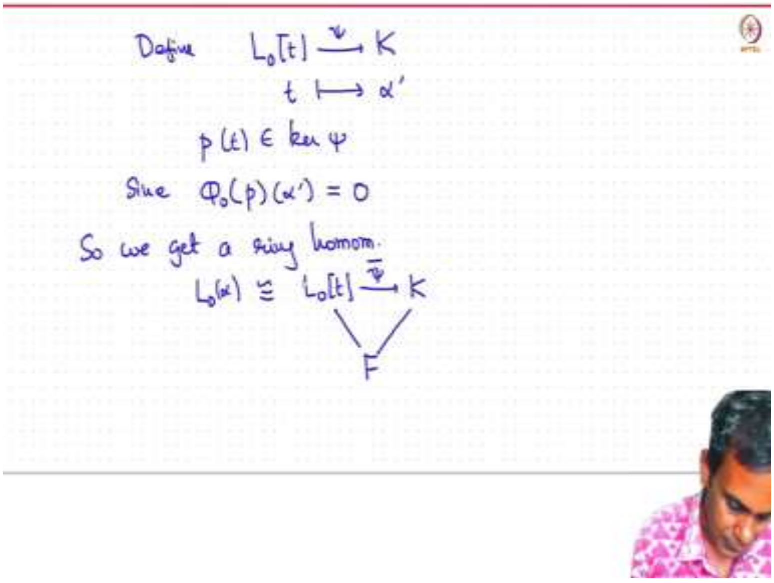

So now what I will do is define a map. I am going to extend; I am going to define a map from L0 alpha into K. So firstly, what I will do is I will take L0t to K by taking t to alpha prime. Now it follows that phi0t belongs to kernel of psi, because psi of phi0 will take the coefficients of the polynomial to the corresponding, the image under phi0 in K and alpha prime has been chosen.

So, it is going to be phi0, sorry, I do not mean phi0t. I mean Pt is in the kernel of psi. Since phi0P of alpha prime is equal to 0. We have chosen alpha prime to be the root, a root of phi0P. And so, the psi gives rise to a ring homomorphism psi bar L0t to K. And this extends phi0 but phi0 restricted to F is the identity of F. So, this map psi bar restricted to F is also the identity of F.

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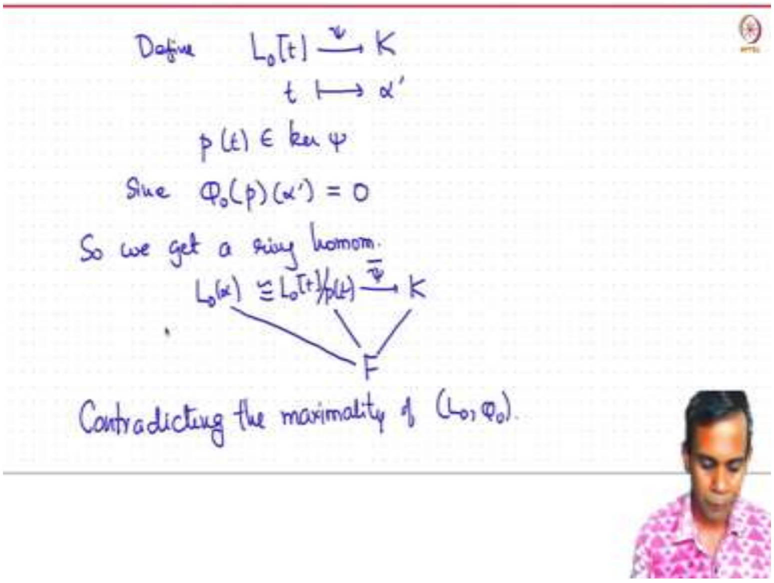

On the other hand, what we have is L0t is isomorphic to L0alpha, as is always the case because alpha is a root of not L0t, this is L0t mod Pt. L0t mod Pt is isomorphic to L0alpha. And so, what we get is a ring homomorphism from L0alpha to K and this of course is you know reserves is is when restricted to L0, it is the identity of L0. So, certainly when restricted to F it is still the identity of F.

And so, we get a ring homomorphism by composing this isomorphism with psi bar, we get a ring homomorphism from L0alpha to K over F. And this is exactly; this contradicts the maximality of L0 comma phi0, because we have extended it to a larger field L0alpha properly containing L0. And so, that concludes the proof of the lemma and as explained also the proof of the uniqueness of an algebraic closure.