Algebra – II
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Lecture 1
Algebraic and Transcendental Numbers

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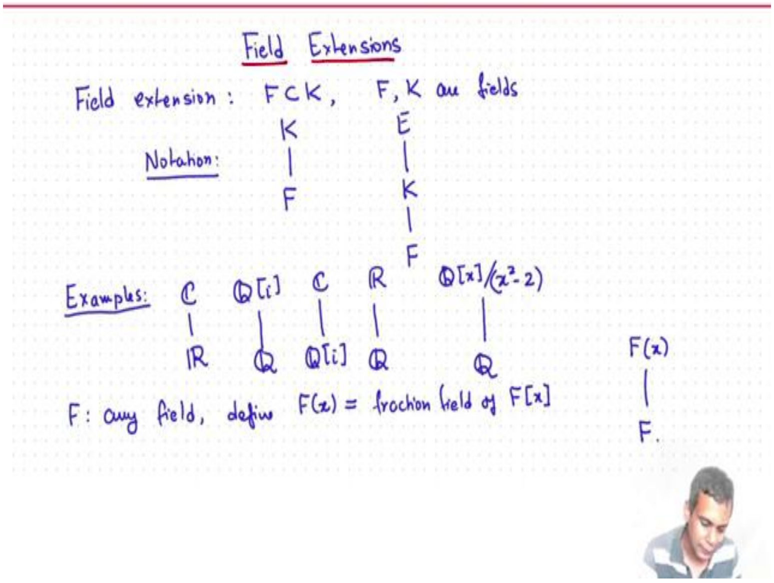

In this part of the course, we will study field extensions. So, what is a field extension? Field extension is just a pair of fields, one is contained in the other. It is a pair F containing K, pair F and K are fields. Usually, we write this field extension, in the following visual form, we write the big field on top, and we write the smaller field on the bottom, and we link them by a line. So, a line going from up to down means that the lower end of the line is a subfield of the field on top. We may also have our towers of fields like a big field E containing a field K, containing field F. And we will see more complicated diagrams of this kind later on. This is just the notation for a field extension.

Let us look at some examples. We have already seen many examples of fields. So, from these, we can construct examples of field extensions. So, the complex numbers, the field of complex numbers, contains the field of real numbers. The field of Gaussian numbers contains the field Q. So, this remember, this is just all complex numbers will form A plus IB, where A and B are rational numbers.

And then you could talk about the field of complex numbers. But that is an extension also in the field of Gaussian numbers, it contains the Gaussian numbers. You can talk about real numbers contains Q, that is another example of a field extension. And here is another interesting example. So, you just take Q of x, mod x squared minus 2. So now x squared minus 2 is actually irreducible polynomial in the ring Q of x. And so, Q of x mod x squared minus 2 turns out to be a field and this is an extension of Q. Let us look at a slightly more interesting example.

So, given a field F. You can start with any field, define F, round bracket x to be the fraction field of the polynomial ring F x. So, an element of this field is a ratio of 2 polynomials with coefficients in F in the variable x. And this is a field and this contains the field F. So here is another example of a field extension F x contains F. More generally, we can talk about multivariable fraction fields.

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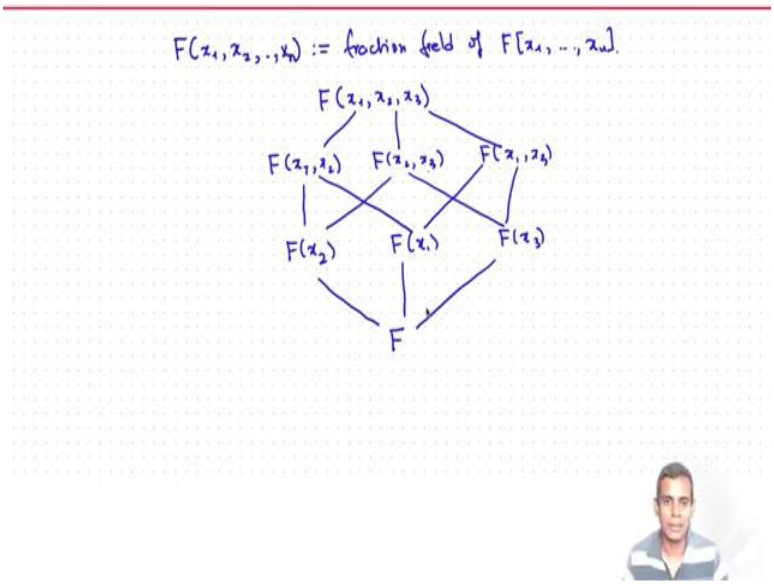

So, for example, you can talk about F round brackets x1, x2, and so on. And this is to defined as the fraction field of well maybe we have n variables here of the polynomial ring in n variables. So, now, you can see some interesting. So, you have the field x1, x2, x3 rational functions in 3 variables. This will contain fields of rational functions in 2 variables x1, x2, F of x2, x3, F of x1, x3. And then here you have F of x2, F of x1, F of x3, the fields of rational function in one variable or F of x2 is contained in F of x2, x3 and F of x1, x3, F of x1 is contained in F of x1 x2 and F of x1, x3, F of x2, x3.

And then finally we have F at the bottom. So, this kind of diagram shows the relationship between several fields at the same time. So here we have 6, well, no 8 fields. And this diagram tells you which ones are extensions of which ones. So, this shows you the power of the notation that I introduced last time for field extensions.

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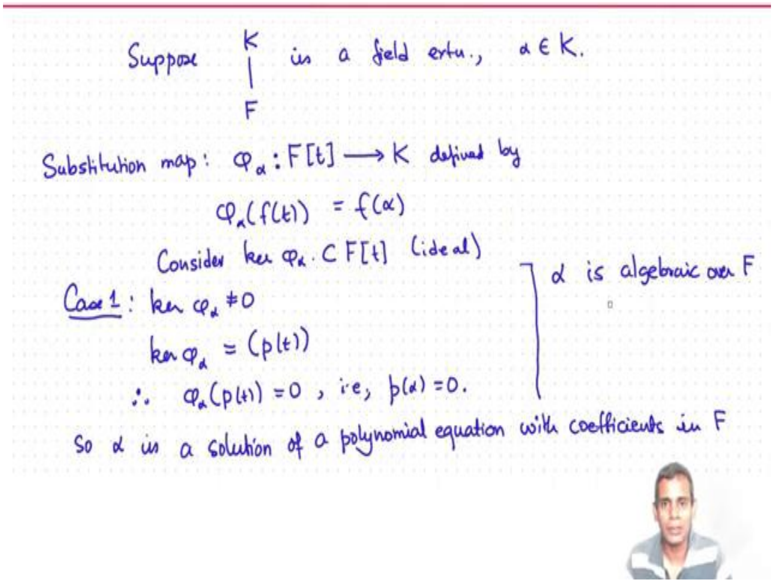

Now, suppose we have a field extension. And we have an element alpha in K, then there is substitution map, this is a ring homomorphism from the polynomial ring F, maybe I should call it t to K defined by. So, this substitution map is let us give it a name, we will call it phi subscript alpha, defined by phi subscript alpha of say some polynomial F t is the polynomial F evaluated at alpha. So, this is a ring homomorphism. It is not difficult to see that this is a ring homomorphism from the ring of polynomials F t to the field K.

Now, consider the kernel of phi alpha. And there are 2 cases, case 1 the kernel of phi alpha is non-zero. Now remember the kernel of ring homomorphism is an ideal in F t and F t is our principle ideal domain. So, kernel of phi alpha is a principle ideal, kernel of phi alpha is the ideal generated by a single polynomial p t . This means that p of so phi alpha of p is 0, i.e., p of alpha is 0.

So, what this means is that alpha is a solution to a polynomial equation where the polynomial has coefficients in F. So, alpha is the solution, a solution of polynomial equation with coefficients in F. In this situation we say that alpha is algebraic over F. So, we say that an element alpha of K is algebraic over F, if it is the solution of a polynomial equation with coefficients in F.

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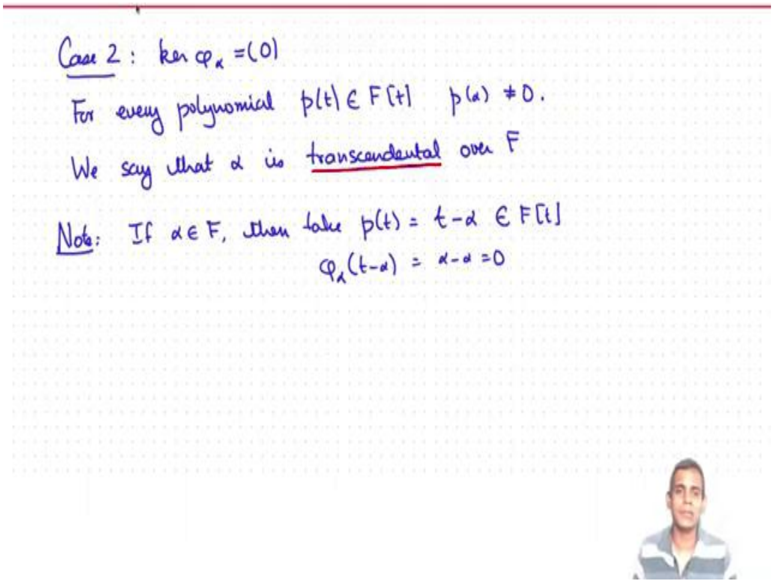

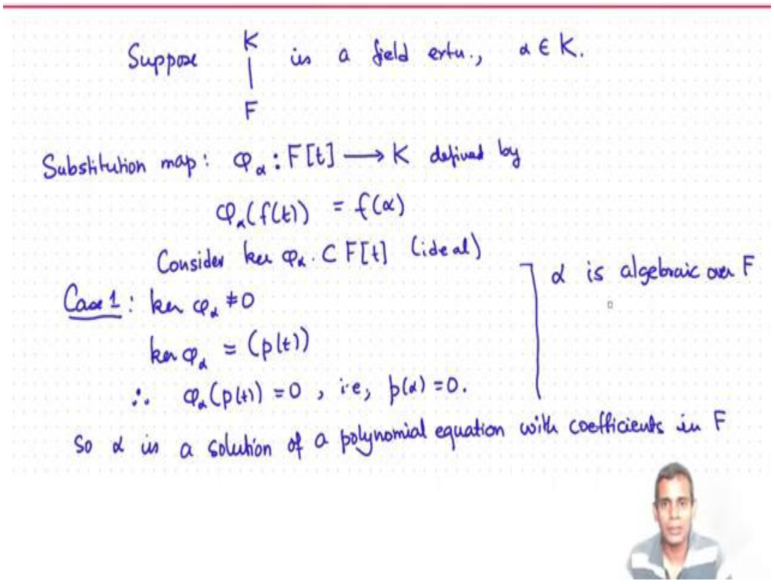

The second case is where kernel of phi alpha is trivial. In this case for every polynomial p t. So, we are looking at polynomials with coefficients in F for every polynomial p t in F t, p of alpha is non-zero. In other words, the substitution map is injective. In this case, we say that alpha is transcendental over F. So, we have 2 kinds of elements. In a field extension, either an element is algebraic or it is transcendental.

Now note right off one simple fact that if alpha is in the small field F itself, if alpha belongs to F, then take p t to be t minus alpha. That is a polynomial in F t because alpha is an F. And so, we have phi alpha of t minus alpha, well, that is alpha minus alpha, which is 0. So, every element of F is algebraic over F. And so, the only interesting question is when an element is not in F, is it algebraic, or transcendental over F? Let us look at some examples. So, the simplest example may be, would be to look at the complex numbers of the real numbers that is field extension that we are all very familiar with.

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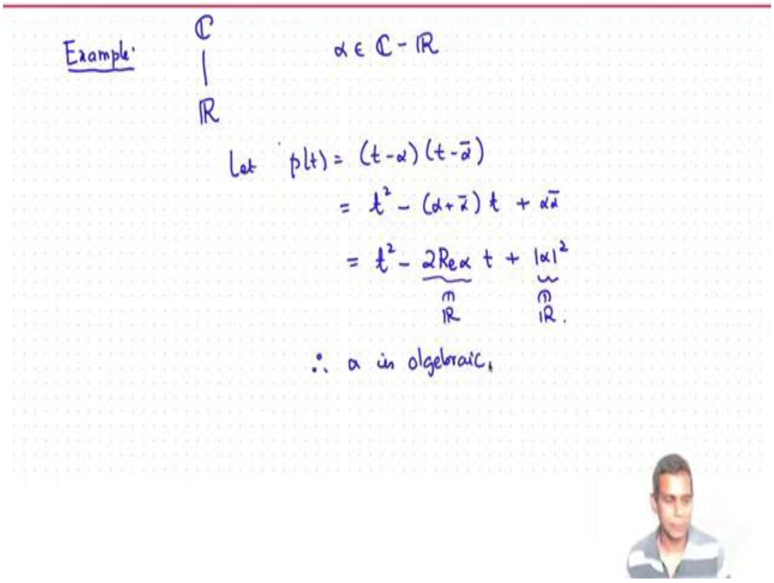

So, look at complex numbers over real numbers. And now we have some alpha belong to the complex numbers, we can assume that alpha is not in the real numbers. So, let us assume that alpha is a complex number, that is not a real number, which means that alpha is not equal to alpha bar. And is this alpha going to be algebraic, or transcendental? Well, I claim that alpha is algebraic, we will find a polynomial with real coefficients that alpha satisfies.

And that polynomial is, well, if you are used to solving quadratic equations, you can easily guess what the polynomial is, let p t be the polynomial t minus alpha into t minus alpha bar. Now, this polynomial is t squared minus alpha plus alpha bar t plus alpha alpha bar, which I can also write as t squared minus 2 times the real part of alpha times t, plus the absolute value of alpha squared. So, this is a real number. And this is another real number. So that and alpha obviously satisfies this polynomial p t, because alpha is one of its roots. So, therefore, alpha is algebraic. So, every complex number is algebraic, over the real numbers.

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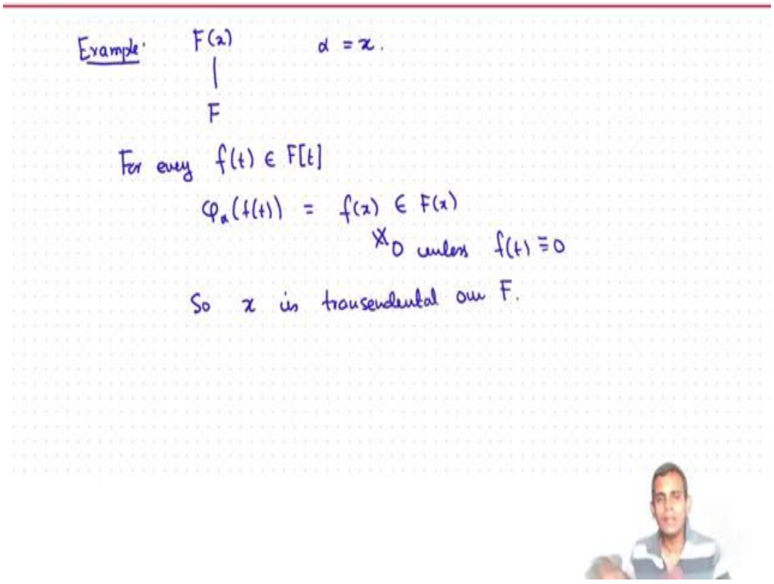

Let us look at another relatively simple example. Let us look at the extension. F x over x, F. And let us take alpha equal to x itself. Then, for every polynomial F t, belonging to F t phi alpha of F t is just you are substituting F of x, x for alpha. And so, this is just an element of F x. And this is not equal to 0 unless F of t is 0 in is identically 0 as a polynomial. And so, x is transcendental over F. Now, let us come to a slightly more interesting and in some sense, still very active area of inquiry.

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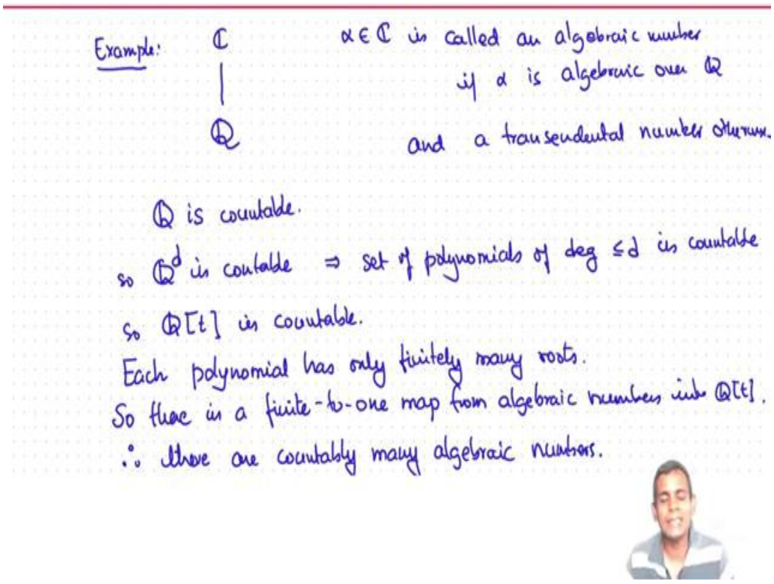

An interesting example, which is to look at complex numbers over rational numbers. And we ask, can we find our transcendental numbers here are there complex numbers which are not algebraic? So, in this context, so if alpha belongs to C is called an algebraic number if alpha is algebraic over Q and it is called a transcendental number otherwise. So given any complex number, you can ask, is it transcendental or is it algebraic?

Now there is a very simple counting argument, which shows that there must be lots of transcendental numbers and that algebraic numbers are in fact, quite rare. In fact, algebraic numbers are countable, the total the set of algebraic numbers is a countable set. But the set of complex numbers has cardinality the continuity (()) (16:09), it is an uncountable set. And so there must be lots of transcendental numbers.

How do you see that? Well, I will just sketch the steps in the proof. So, firstly, we start with the fact that we all probably know that Q there are set of rational numbers is countable. So, that means that Q to the D is countable, which means that polynomials of degree D they have D coefficients each come from Q. So, the set of polynomials of degree less than or equal to D is countable.

Which means that the set of all polynomials with rational coefficients is countable because a countable union of countable sets is again countable. Now, each polynomial has only finitely many roots. So, given any algebraic number, we can associate to it one of the polynomials of which it is a root and this caves of finite to one map from algebraic numbers to polynomials in Q. So, there is a finite to one map from algebraic numbers into a countable set Q t, polynomial in one variable.

Therefore, there are countably many algebraic numbers, and hence, there are uncountably many transcendental numbers because the complement of a countable set in an uncountable set is still uncountable. There are lots and lots of transcendental numbers, but it turns out it is not so easy to prove that a number is transcendental.

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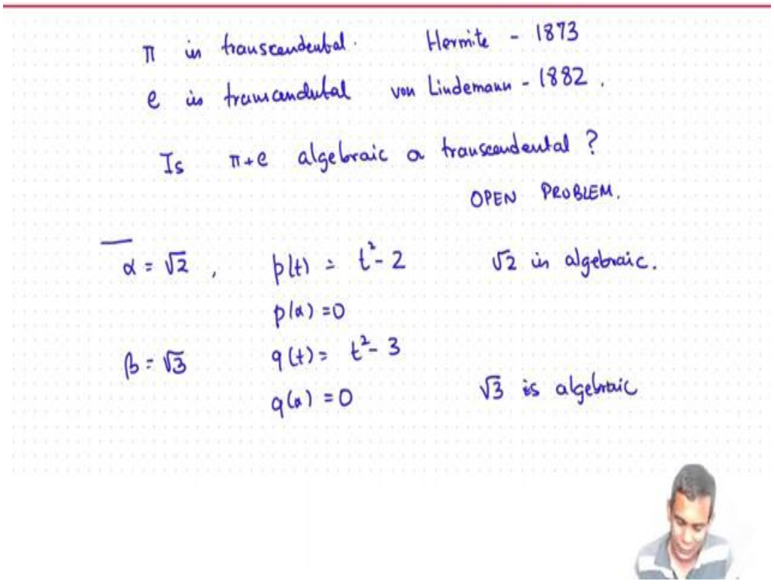

So, the first well-known number to be shown to be transcendental was Pi, Pi is transcendental and this was proved by Hermit in 1873, then Euler's constant E is transcendental. And this was proved by Von Linderman in (()) (19:47) 5 years 1882. But to this day, it is not known whether Pi plus E is algebraic or transcendental. So, while we know that there are lots of transcendental numbers out there, it is difficult to show that a given number is transcendental. And let us look at a more concrete example. So square root 2, so let us say alpha equals square root 2. So, is this algebraic or transcendental?

Well, this one is easy. Because if you take p t equals t squared minus 2, and then you substitute square root 2 here is a solution for this equation, then p alpha equals 0. So square root 2 is algebraic over Q. Hence, it is an algebraic integer, square root 2 is algebraic. What about square root 3? Well, now you can take p t equals t squared minus 3, maybe we will call it Q t, then we have Q alpha equal to 0, so, square root 3 is also algebraic. Well, in the spirit of Pi and E, we can ask is square root 2 plus square root 3 algebraic?

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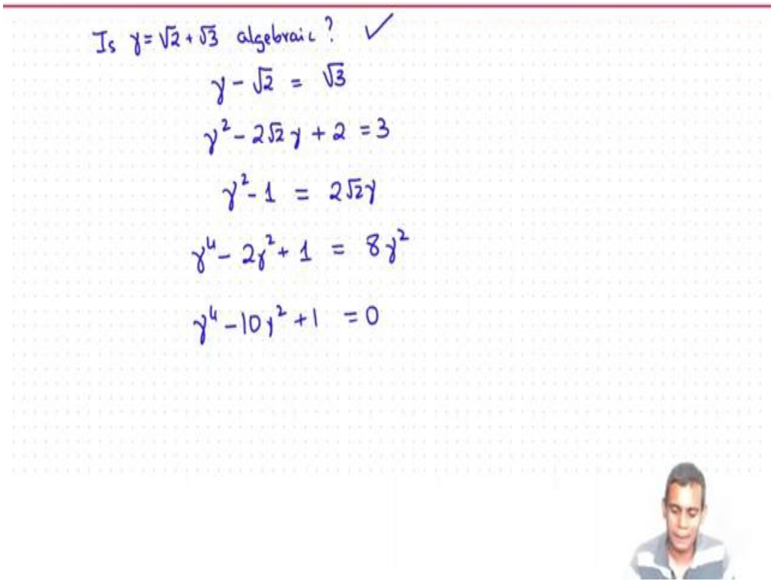

In a later lecture, we will see a very conceptual reason why square root 2 plus square root 3 is algebraic. But right now, let us just try to find a polynomial for which square root 2 plus square root 3, a polynomial with rational coefficients, so with square root 2 plus square root 3 is a solution. So, look at this gamma, we want to find a polynomial such that gamma satisfies this polynomial.

So, gamma minus square root 2 is equal to square root 3, if we square that, we get gamma squared minus 2 root 2, gamma plus 2 is equal to 3, which we can rewrite as gamma squared minus 1 is equal to 2 root 2 gamma, and then we can square that again and we get gamma 4 minus 2 gamma squared plus 1 is 8 gamma squared. And so, we get gamma 4 minus 10 gamma squared plus 1 is equal to 0. So, there you go, there is a polynomial of degree 4 with rational coefficients that are satisfied by square root 2 plus square root 3. So, square root 2 plus square root 3 is also algebraic.