Introduction to Probability – With Examples Using R
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Lecture 8
Sampling and Repeated Trials

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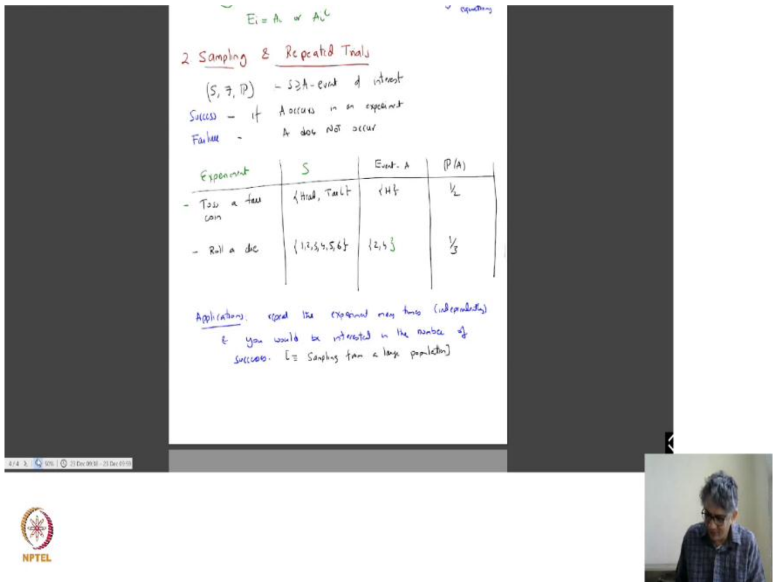

So now let me start off with this following idea that, what, how does one use independence. So, these are the next chapter, second chapter in the book, it is called, it is a very important property in statistics and probability how to sample, it is called sampling and the other thing is that is understanding of independence through repeated trials or the reverse values say, you understand relative frequency to repeat trials and then use the independence property to understand all these things.

Now so let us consider an event A with sample space S. Now so you think of an event A as some especially when they are interested in let us say you are tossing a coin and your event A is getting a head, so you take you have S sample space in the event space and the probability P let us say, so this is A and you have an A is an event of interest and A subset of S so A sub set of S, so S contains A. So now the you say you deem two things you deem a success if A occurs, if A occurs in the experiment and you say define the failure if A does not occur.

So your A is your event of interest and if A does not occur. So, what you do is you will conduct experiment it has many outcomes perhaps, but you are interested only an event A, A is an event of interest and you would define yourself you define your whole thing would be okay, I will say the experiment is success if A happens, experiment is a failure if A does not happen.

So, let us look at a few examples, let me do a few examples, so let us say the experiment, one experiment I have and it says I toss a coin, I toss a fair coin that is my experiment. I could also have an experiment that say let us say I roll a die and that is my experiment. Then let us say I, my sample space I know exactly my sample space S I know exactly.

So, S here is toss of fair coin my sample space is head or tail, that is clear to see. Then if I roll a die the sample space exactly is 1, 2, 3, 4, 5, 6. What is an event? Let us say the event that I want, that is the event A, let us say that I want the event is that a head occurs, let us say and let us say the event A I wanted the roll is let us say an even number happens, so even number happens this is 2, 4 and 6. Then I know the probability of A in each of these cases. So, in each of them are equal likely outcome experiments.

So, each of them so here it is going to be one half, here it is going to be again one half, 2, 4, 6 (()) (4:13). So now maybe I will just change my event let us say 2 and 4 occurs. So, that means the difference (()) (4:20) let us say it is, so only 2 or 4 occurs is my event A of interest and that would just mean the probability of A is one third. So what, you do is you have an experiment there is a certain probability of occurrence.

So, typically what we would do is, this is a probability model that you face for experiment. So in applications what you will do is you do not know what probability of A is. So, to estimate that what you do is you would repeat the experiment many times and independently there is one trial does not affect the outcome of the other trial and you would count, you would be interested in the number of successes and that is what I think. And this is what I mean by this kind of, you think a little bit this notion of repeat experiment many times is kind of sampling from a large population.

We will come to this later on in the class why this is the same thing. You could intuitively think the following let us say you are a factory that makes pens let us say and you want to know if your pen is perfect all the time, your machine is working well. What you do is you go, you produce a batch of 100 you pick one, it is one pair at random and see if it is correct, if it is well done you call it, call that batch success; if it is not done you call the batch a failure.

So then there is a chance you just see how many times this happens. That gives you how well your machine is working and so on so forth, and you repeat the experiment many-many times to get an idea of what the proportion of defects are in your machine. So, that is the, that is the broad out, broad motivation for doing an experiment many-many times, this is called sampling of repeated trials.

So what I do now is I will proceed to give you a mathematical framework for repeated trials. It does (()) (7:02) with the applications part. But all you know is that the motivation is that you, you have an experiment that you repeat independently many-many times and you count the number of successes to estimate the probability of success because the relative frequency of success is the probability in an experiment that you conduct. But in terms of a model I will try and tell you how to compute probabilities if you repeat experiment many-many times. Very good.

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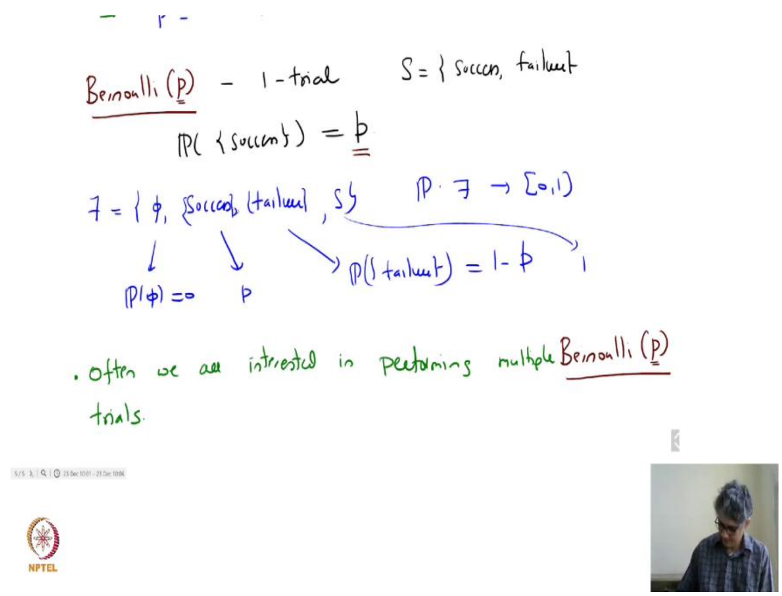

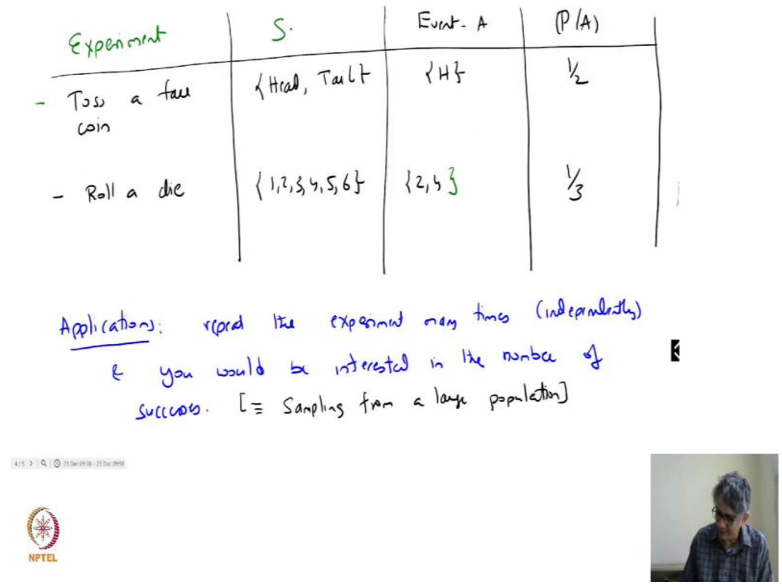

So, these are called Bernoulli Trials. So, (()) (7:46) math (()) (7:47) in high school you may have seen there is a family the French sort of I think a bunch of people in the family named after Bernoulli's last name were all mathematicians and James Bernoulli is sort of created with this idea of Bernoulli Trials. It is around the 17th century. So, what, what here is that this this provides a mathematical framework so this what I am going to do is I am going to provide a mathematical framework, framework for independent trials of an experiment.

And what I am going to do is I will designate. So, what we will do is we will just use erase this little bit. So, I perform experiment so I has many outcomes but I am interested in same event A so all I do is I will designate the outcomes has two parts a success or failure. So then you could think of an experiment having just two outcomes that means one is success, one is failure or you could think of having an experiment many outcomes but your event of interest is A and if A happens you call it success, if A does not happen you call it failure.

And what I will do is I will just say I have, I know the probability of success like the previous the table that I drew for you. So, I will just assume that P is the probability of success. Whatever success is defined at each trial. So I stop. So, now I could forget about my experiment, I could forget about my largest sample space, I could just fix my sample space to be a success and failure and work with that. So, this is what Bernoulli did, this Bernoulli trials. So, it is called this is called Bernoulli P, Bernoulli P. So, here I would just assume that instead of many trials I will just say it is one trial.

You form a trial, so your trial happens, you have a success or a failure. So, sample space S is just a success happens or failure happens and we know that the probability of success is P. So, this is clear. So suppose I taken, I take a, why is this enough to discuss the whole sample space because I know several things. If I have S is given by this then I know my event space is just empty set success, failure, the set outcome failure, the outcome success and the outcome S that is my events space.

Because subset of all sub sets has only these four guys and my P is automatically defined from f to 0 1 because the moment I know success, I know several things, because I know this is automatically a probability of empty set is 0 that I know. This I know is P by definition, this is, failure is the complement of success so probability of failure is going to be 1 minus the probability of success which is just 1 minus P and this we know is just 1.

So immediately probability also is very different. So, Bernoulli P trials means that the sample space is success or failure and the probability success is P. That is the P here. So, suppose I say I have a Bernoulli P trial that means I perform one experiment where the probability of success is P and the experiment has only two outcomes success and failure.

But often we are interested in how to do multiple trials. But often we are interested in performing multiple trials, in performing multiple Bernoulli P trials. So, I will just call it Bernoulli P trials, which is we understand. That is, we perform an experiment again and again, and each time the probability successes is P. So, let us see how to understand this. Let us see how does one work with it. I perform an experiment many-many times independent of each other and experiment have only two outcomes success or failure and I want to know how to perform computations with them. So, let us do this.

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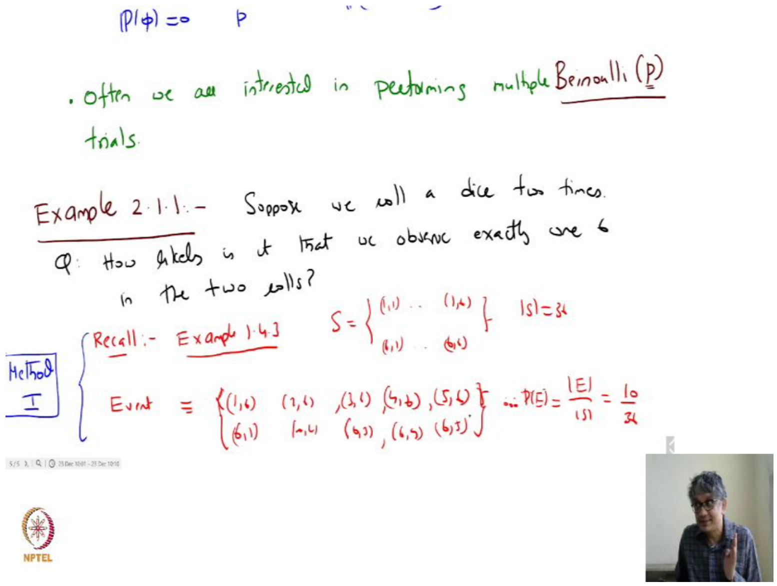

Let us call this example 2.1.1. Suppose we roll a die twice, so, I will discuss an example where this sort of Bernoulli trials occur naturally, two times. And my event A of interest, so I am interested in let us say, I will not phrase it like the event A, I will say I am interested in understanding I am asking a question, how likely is it that we observe exactly 1 6 in the two trial, in the two, in the two rolls?

So, this idea if you recall, we did this example before long time back. So, let us say we recall, it is a recall in example I think it was 1.4.3. So there what I did? I said okay I have a die, I have rolled the die two times, I wrote down S as my 36 outcomes over there and 1 1, all the way to 1 6 and then add our 6 1, all the way to 6 6, and 1 S is equal to 36 and I had model reflecting equal likely outcome experiment.

And I, I describe my event of interest as I am doing only 1 6, so I say that event of interest is event is given by only 1 6 that seems like 1 6, 2 6, 3 6, 4 6, 5 6, as the event of interest another one, 6 1, 6 2, 6 3, and so on 6 5, only 1 6, these are all the all the possible outcomes I am interested in and I would do that the probability is, is mod E by mod S and the answer would be, answer would be here it is just ten people 1, 2, 3, 4, 5; 1, 2, 3, 4, 5, 10 by 36.

So, that is what I did before, that is what I did. But now I am, this is some little cumbersome this method is my let us call this method 1. So, the advantage of method 1 is the following that you are able to prescribe the entire sample space properly, you understand like an equal likely experiment, you discuss your event, event E of interest in a clear way and then proceed accordingly. But now I do not want to do this, I want to use the fact that each roll is independent and somehow can I use that to compute the question of interest. So, what I do is I am only concerned with 6 occurring or not occurring.

So, what I do now is I see this method 2, method 1 is this that I list the outcomes out, I think of it as an equal likely experiment and I move on. But I cannot keep doing that again and again. If I repeat these ten times then I do not want to keep doing this, write down whole 10 tuples and a huge, huge space and it is just too much.

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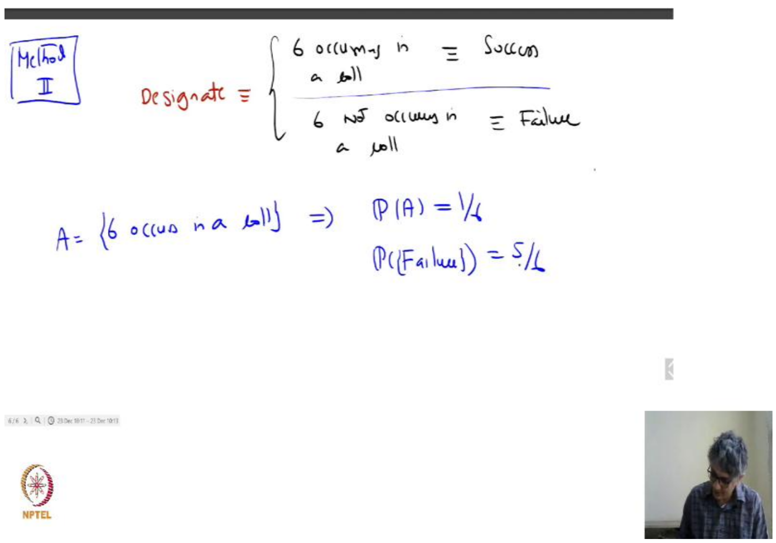

So, this is method 2. So, I will try and describe it today and then I will build on this when I come next week. So, what happens is that here I am only taking 6 so what I do is I designate the outcome 6 as success. So, I will say let us say designate a 6 occurring in a roll as success and 6 not occurring as failure, in a roll as failure. So, this is what I do, I do this terminology I say fine. So, I designate the fact that if 6 happens I call it a success. If 6 does not happen I call it a failure.

So, when I do a roll, I am converting it to a success and failure experiment in this formula. So, that is how I do, very good. Now I do the following, now I know already that for every single roll it is easy to do, I have single roll, probability of success is, let us say A is the event I am interested in that 6 occurs that is in a row.

This I know, we will imply that probability of A is just 1 6th because in an event that there is a 1 through 5, 1 through 6 outcomes possible equal likely and probability of A is just 1 6th, and A complement is probability of success and probability of failure as A does not occur is just 5 6^{th} that is also easy to say, very good (()) (19:55).

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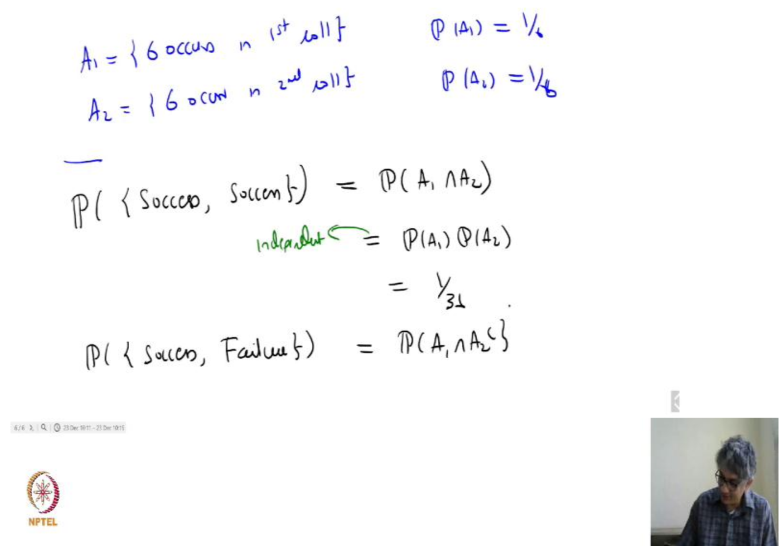

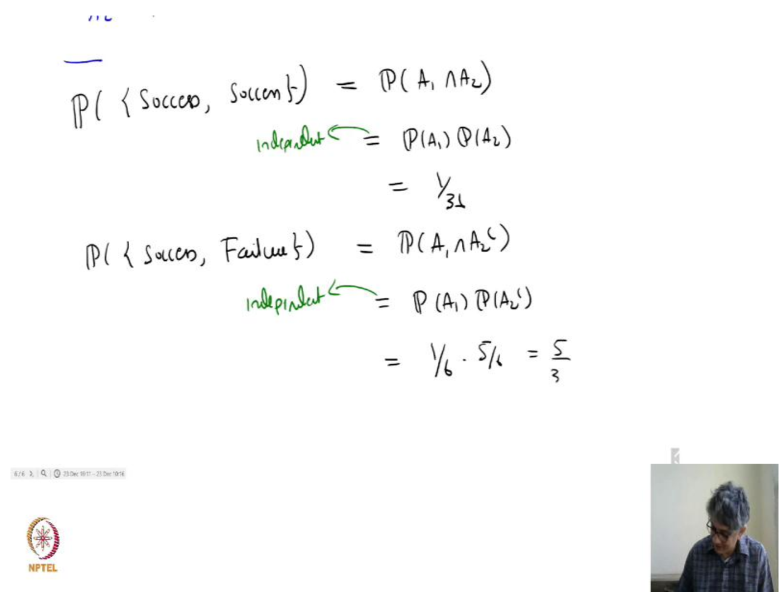

So, now what I do is I go into the following, this is my event, so what I do is I say A1 be 6 occurs in the first row, what happened now, 6 occurs in the first row, and I say A2 is the event that 6 occurs in the second row, very good, I know this term. And what am I interested in, I am interested in? I am interested in 6 occurring in only one row. So, now this is what is, this is same as saying I already know probability of A1 is 1 6th probability A2 is also 1 6th. There is no sign here for sure. Pretty (()) (20:51) is independent and this consists only one roll.

So, now I do the following, I remodel this whole thing as a success failure experiment that is what I am trying to do. So, how I do is I look at this, I look at probability I have two trials now. So, one of one possibility is success and success. The each roll, I have success in both dice, is the same as probability of A1 and A2 we have already seen in the example I did today that A1, A2 are independent, seen earlier that A1, A2 are independent.

Use independence to get probability of A1 times probability of A2 and that is the same as 1 over 36. Now I also do the form, I do probability of success and failure. What they are going to be, let us call it probability A1 and A2 complement because that is what, that is what failure is. Failure means I do not have A2 I do not have 6 in second roll.

That is same as again independent. So probability A1 times probability A2 complement. So, again in that shown earlier because we know that A1, A2 are independent then A1, A2 complement also independent. That is going to be equal again 1 6th for A1. What is A2 complement? Probability of A1 is this, A2 is this, probability of A2 complement is just going to be 5 6th. There is one roll, this is going to be 5 6th. So, I get 5 over 36, very nice. Then probability of, then I will do the same thing I can just let me just since I am on I-pad I am going to do this way.

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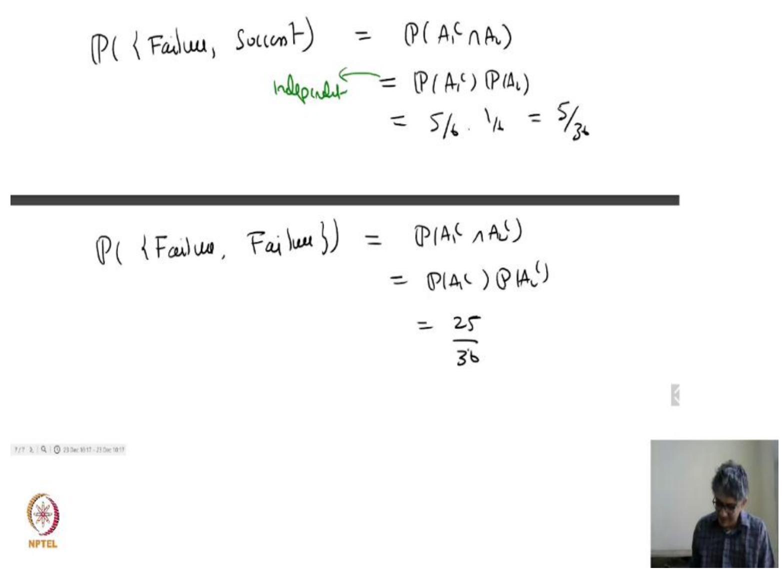

So, probability of failure and success. Let us fail in the first one, is same as A1 complement and A2 then again by independence, independence will be same as probability of A1 complement times probability of A2 that is the same as again 5 6th times 1 6th and that is just 5 over 36, very nice. And similarly to complete the experiment I can just say that probability of failure and failure, so that is the last (probability) last event of the samples we have to consider, last oncoming samples that consider failure-failure and that is the same as probability of A1 complement and A2 complement and that is again the same as probability A1 complement times four A2 complement and that is going to be equal to 25 by 36.

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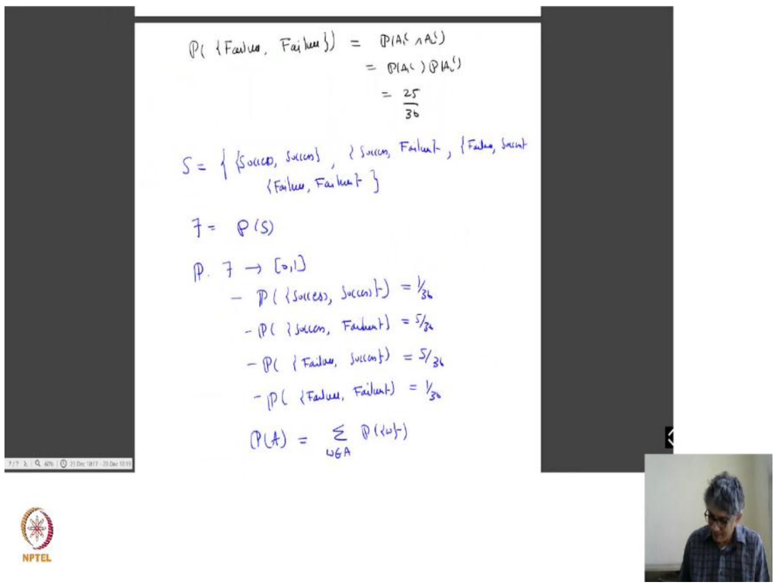

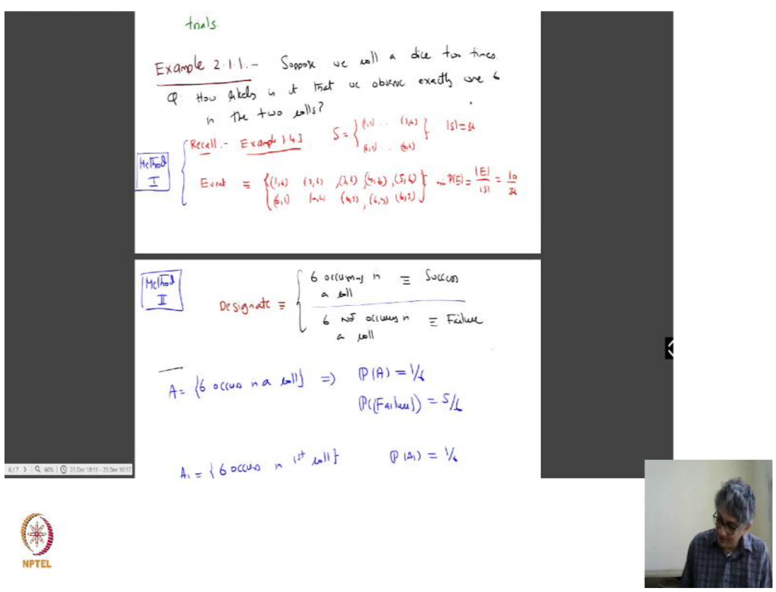

So, in a sense in a sense I have converted my two rolls because of my equation of interest I have converted my two experiment into experiment with sample space S as success, success; success, failure and failure, success and failure, failure. This is what I did, event space is just (()) (24:59) of all this, anything you want, you take foundation and I know that my function P is F from 0 1 and I identified for every outcome I know the event that success, success is 1 over 36.

The probability of success, failure that I computed above was 5 over 36 and the probability of failure, success that I computed above was again 5 over 36 and the probability of failure, failure was again 1 over 36 and that is enough to give probability because once I know that the probability of every outcome the probability any other event A we know that let us say any other event A is just going to be the sum over all omega in A probability (()) (26:06).

That is how I define a product. So, then once I did just a side diversion of how I define my sample space but let me get back to the question of interest what is the question of interest? Question of interest was we are exactly 1 6 in two rules. So, that in this corresponding sample space is just, in this corresponding sample space is just one success and one failure or one failure and success.

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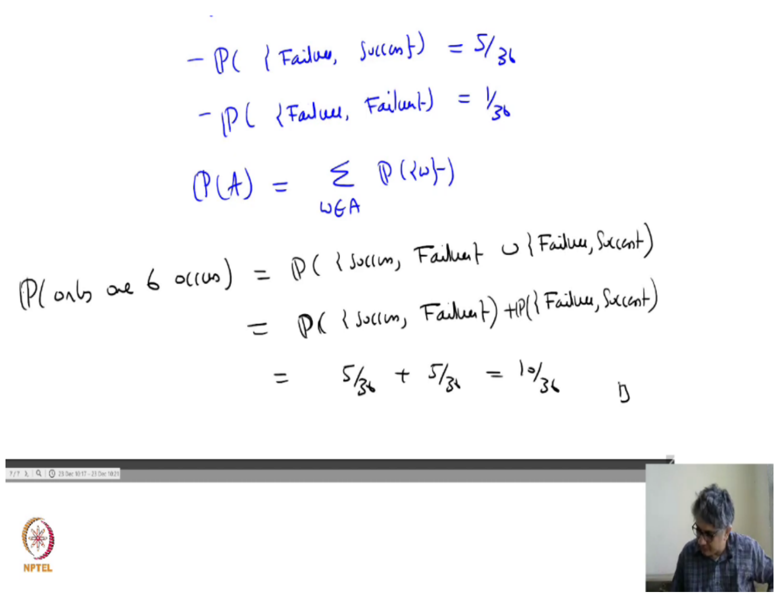

So, that means probability only once 6 occurs is same as probability of, excuse me, success, failure or is the probability of only 1 6th happening, occurred which one interest is the probability of success, failure plus probability of sorry let me write the first union or failure, success that mean only 1 6 has happened and that is the same as the probability of each of them separately. Because they are mutually exclusive and we know each of these is 5 over 36 plus 5 over 36 which is 10 over 36. So, it is another way of doing an example, so let me just quickly recap for a little bit so that you are on the same page.

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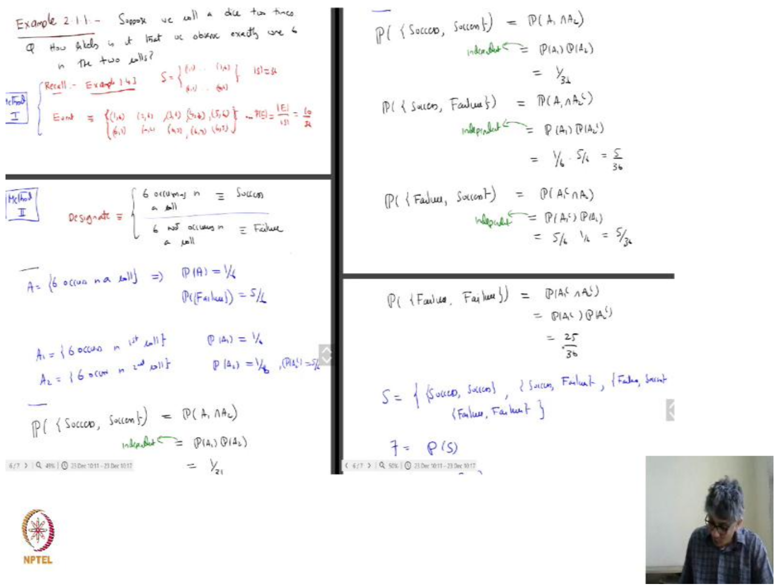

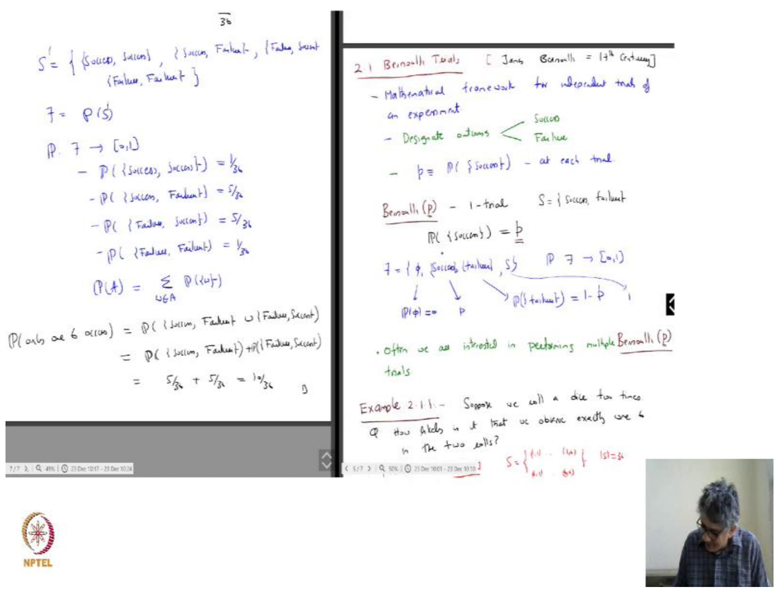

So, the idea was the following that I had this question that I had in mind. I had I rolled it out two times and then I am interested in the fact that only 1 6 happened with 2. So, method 1 we discussed many times before, some days before sorry, in which what I did was I laid out the entire outcome space as 36 possible outcomes and I then I wrote all my event of interest which is

1 6 2 6 2 5 6 or 6 1 6 2 3 6 5 there are 10 elements there. So equal likely experiment, so the probability of E was 10 by 36.

So, this I could do with two quite easily but suppose you can easily immediately see that you will get quite complicated if I want to do let us say 10 rolls and my event was a little different let us say event was let us say an even number occurs, it being quite complicated write down the event and write down that. So, I need a different method to handle this which I will see, which I will discuss next.

So, is method 2. So, method 2 was the following, so what I did was I said, okay 6 happens I call it a success, if 6 does not happen I call it a failure. So, then what happened was then I know that chance of us failure is 5 6th, chance of success is 1 6th, that is kind of easy to see, because you just rolling a die once, you know if 6 happens is 1 of 6 possibilities, if 6 does not happen it is 5 out of 6 possibilities; it is 1 6th and 5 6th.

So, then I am in business, then I look at I look at the fact that I have independent trials. So, if I want to compute success and success I go and look I say okay fine I have probability of A1 and A2. That means I have independent events so that is same as probability A2 times A2. So, immediately I get 1 over 36. So the advantage here was I use the fact about one trial to get what happens to a two trial event. Similarly, success, failure I can do the same way, you know it is just A1 complement and A2 and A1, A2 complement again by independence I get 5 over 36, independent account for 36 and (()) (30:57).

So, what I have done is I have converted my equal likely outcome experiment so we should call this S prime (()) (31:05) the same event as before. In some sense same outcomes spaces before what I have done is I have catered to my question of interest and I have converted my sample space into just success and failure and then I know the chance of each success is 1 over 36, (( )) (31:29) and so on and so forth. And all are (()) (31:32), from that I am able to compute the probability of volume 1 6 occurring because that is one (()) (31:37) success, failure and failure, success or failure success and I get the answer.

So, in some sense what I did was I will explain this next time I converted the experiment rolling I have two times into two independent Bernoulli trials with P equal to 1 6th. So, I will come and discuss this idea next time because that is all I did, so if you look carefully at what I did, I just converted my in method 2 I converted my question, the focus of my question I converted my experiment into a Bernoulli 1 6th trial because the probability of success was 1 6th, so that was the point of view.