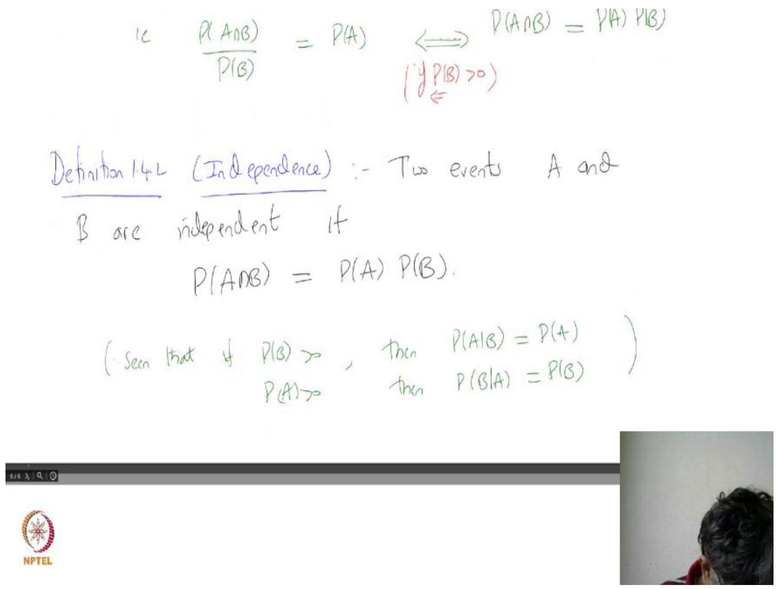
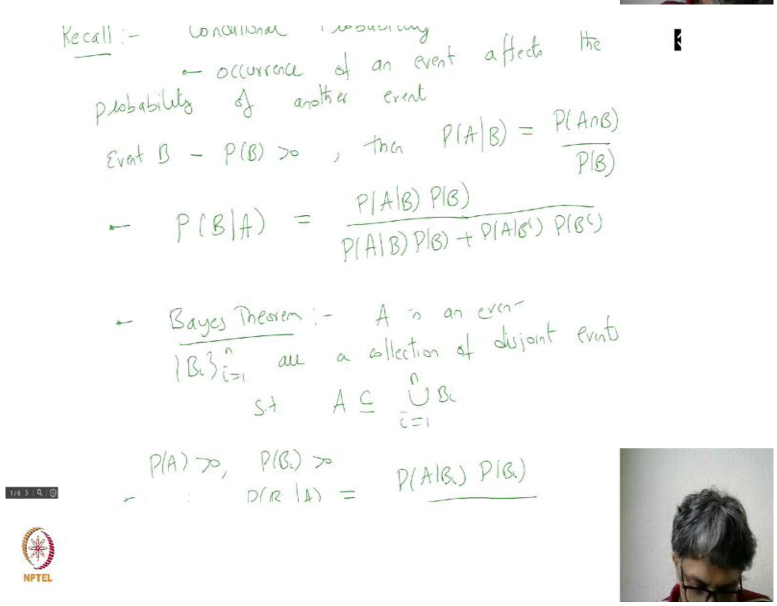
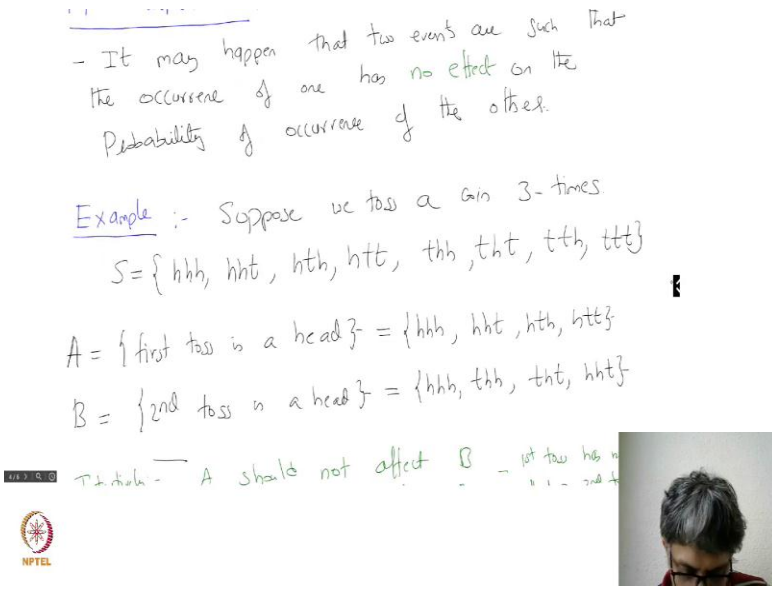
Introduction to Probability – With Example Using R
Professor Siva Athreya
Theoretical Statistics & Mathematics Division
Indian Statistical Institute Bangalore
Lecture No 7
Independence – Part 02

(Refer Slide Time: 00:15)





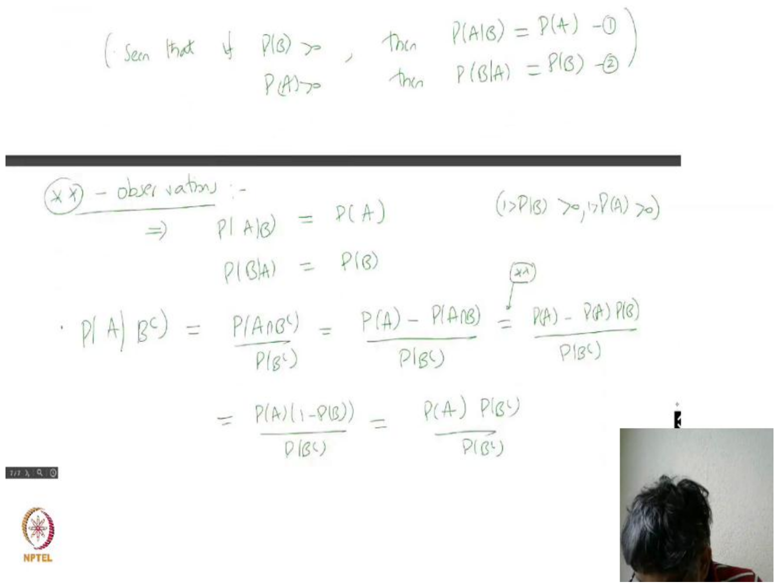


So, we were doing independence and what we will do is, we had done this definition of independence as two events A and B independent, if probability of A and B is same as probability of A times probability of B. And the idea was, this is different from the previous setup where we did last time was this, where one event had a affect on the probability of another event given by this conditional probability formulae.

But here and we discussed this for your own scheme quickly. But here what happens is that two events such that the occurrence of one has no effect on the problem. So, we saw one example where we looked at the first toss in the second toss being ahead and we saw immediately that the probability of A given B was same as probability of A and probability B, given A is here. This led us to the notion of independence, which is probability of A given B is the same as probability A.

But that we write it as in this form, probability of A and B is same as probability of A (()) (1:25). So, let me just comment a bit on the definition of 1.4.2. So, let me just comment on this a little bit. So, there what about a comment on is that, what does this guarantee? So, it guarantees two things, so what is star guarantee let us see. Let us see importance of star. Let us call this a star, double star.

(Refer Slide Time: 01:53)

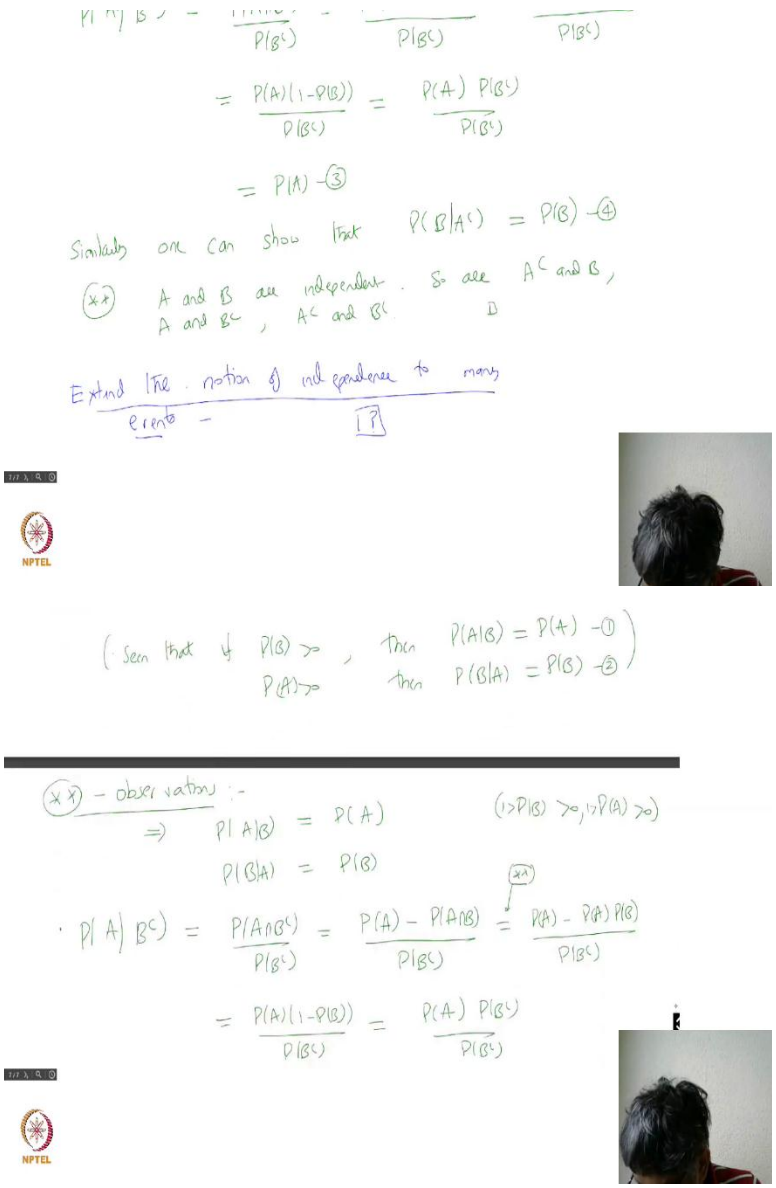


So, so, double star observation. So, the (()) (2:02) double star the following that okay, first we observed clearly that star clearly implies that chance of a given B is the same as chance of A. So, this provided probability of B is positive and then similarly B given A is the same as probability of B. This provided probability of A is positive. So, let us assume these two things. Let us assume that probability of B and probability of A are both positive. So they are meaningful events to understand.

So, now what else does it give you? So, now if you look at the chance of probability of A and B complement, given B complement. So, let us say this is all less than 1 so everybody is less than 1 so all events A and A complement, B and B complement all occur with good probability. So, now here we think of this as probability of A and B complement divided by probability of B complement you need the bottom as it is.

The top person you can use that the properties of probability we have shown same as probability of A minus probability of A and B, divided by probability of B complement, and that is the same as by double star, same as probability of A minus probability of A times probability of B divided by probability of B complement. And if you read at the algebra that is the same as probability of A into 1 minus the probability of B, the whole thing divided by probability of B complement and that is the same as probability of A times probability of B complement divided by probability B complement and we get same as probability A. So, in some sense the moment I know B does not affect the occurrence of A, B complement also does not affect.

(Refer Slide Time: 4:31)

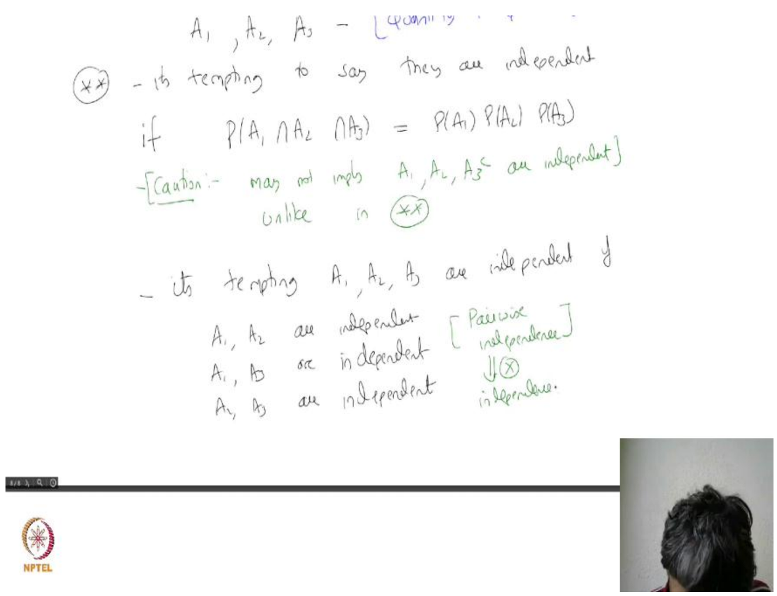


So, similarly one can show, that probability of B given A complement is also same as probability of B. So, in some sense double star if I call this as, as one result equal to double star. One, two result from double star, this is a third result from double star and fourth result from double star. So, in some sense double star contains four equations that is probability of A and B, A given B is probability of A, probability of B given A is probability of B, probability of A given B complement is probability of A and probability A B given A complement is also probability.

So, in some sense, so, double star actually implies all the following that A and B are independent by definition. But, so are A and B complement, so are A and A complement and B, and A and B complement, and one more right and A complement and B component. This I have not checked but it will also imply this. So this, this one kind of funky thing about one equation.

So, in the two case double star implies what you intuitively understand. A does not affect B then B does not affect A, B complement should not affect A and vice versa. So, why is this important? The idea is that, this is important because suppose we wish to extend the notion of independence to many events. Extend the notion of independence to many events rather than just two events. How do we do this? So this is one thing I would like to understand right now and I will try to give an example (()) (6:52).

(Refer Slide Time: 06:58)



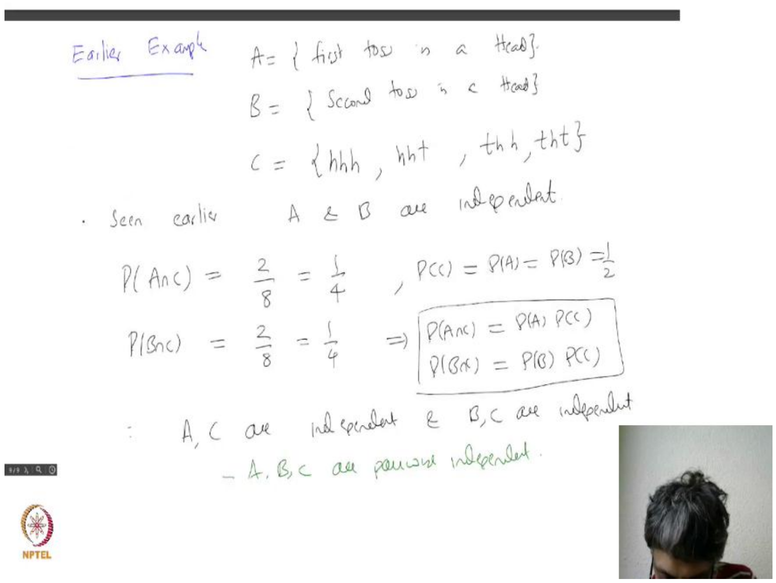
So, let us do three first before we come to, let us just do three, before we come to this idea. Suppose, we have three events, three events A1, A2, A3. So, one thing it is to say let us say and we want to quantify independence, quantify independence. So, one thing is tempting. So, from double star it is tempting to say they are independent if probability of A1 and A2 and A3 is the same as probability of A1 times probability of A2 times probability of A3, so this is tempting to (()) (8:38).

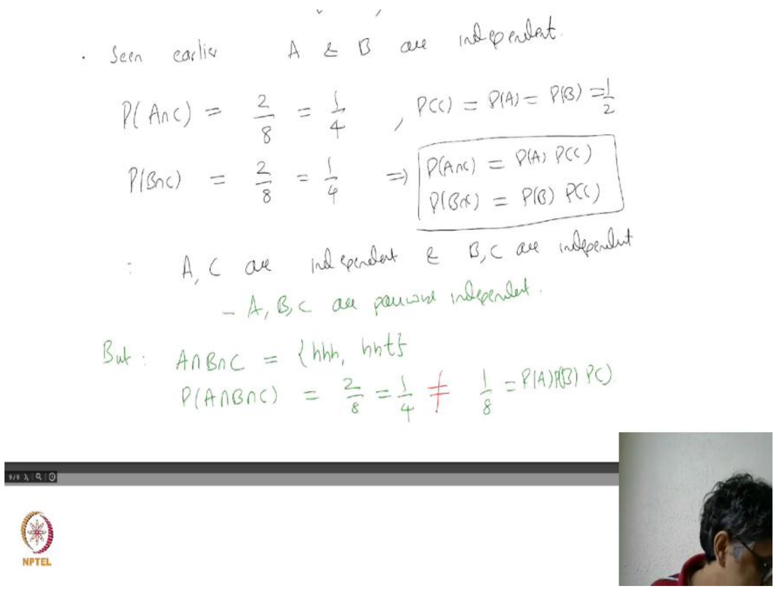
But you should be careful now, this in the two case we got lucky. In the three case, this is a cautionary note. This is one way of writing independence for three people. But this this may not imply, A1 let us say A2 and A3 complement are independent. In the two case, somehow we had this notion and you could do the algebra properly and get it unlike the two case. Unlike in double star where this actually implied.

So be a little bit careful now, you cannot just take this. The second this one is approach A, the other approach is, it is also tempting to say the following, that okay it is tempting to say, each pair is independent. So, let us say it is tempting to say, A1, A2, A3 are independent, if you do not want to say this independent, if A1 and A2 are independent, A1 and A3 are independent and A2 and A3 are independent.

So, that is, sorry, this is something that one could think about but its again has a cautionary tale. So, let us see the cautionary tale, this is like again. So, it is known that pairwise independence is called pairwise independence. You assume that every pair is independent. So, this typically does not imply independence. Let us see the first one, I will let you see an example, this notion let me try and see if I can give you an example.

(Refer Slide Time: 11:13)





So, let us do a simple example. Let us go back to our earlier one, the earlier example. Let us go back from our earlier example. So, A was the, A was first toss, sorry. A was the event that the first toss is a head. B was the event the second toss is a head. And now let us have a C. I introduce an event C that, let us say, I have either head, head in the first toss or tail, tail in the first toss or tail, head, let us say tail, head first two tosses. Let us say I take the event C like this.

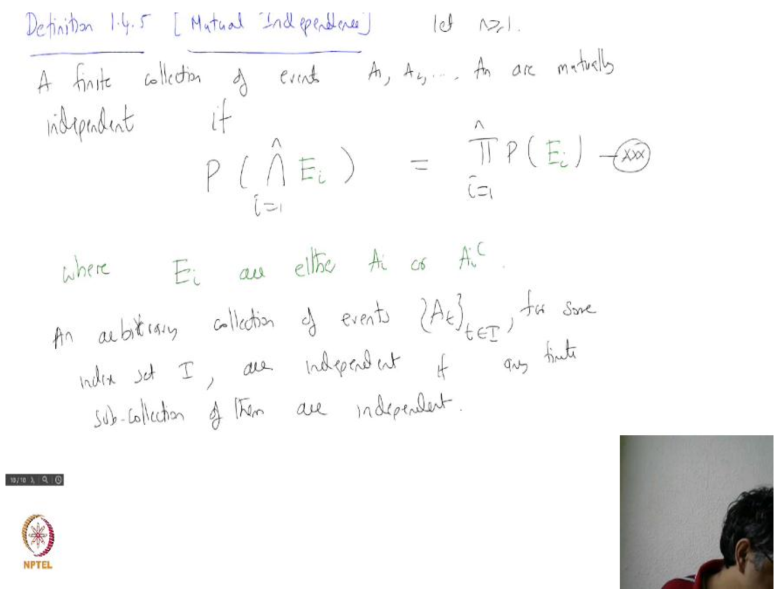
Now we have seen earlier A and B are independent. Now let us try into the calculation. Let us try and do probability of A and C that is the same as the A and C put together. You can just check it is A and C means head in the first toss. So, that is just going to be 2 by 8 and that says 1 by 4 because only two people are common. Probability of B and C is the same again as 2 by 8 which is 1 by 4 and you know that the probability of C is probability of A is probability of B is all half.

So, this will imply that probability of A and C is same as probability of A times probability of C, probability of B and C the same as probability of B times probability of C. That is just the fact. So, therefore A, C are independent and B comma C are also independent. So, they are pairwise independent. So that means we have seen an example, that is an example where A, B and C are pairwise independent.

But known before the following happens. But, what happened now unfortunately if you look at the event A and B and C that means you have a head in the first toss, head in the second toss, so you are left only with two guys now. A and B and C is just head, head, head and head, head, tail. That is just the way it is defined, that is the way it comes out. So, probability of A and B and C is going to be 2 by 8 which is one quarter, and unfortunately this is not equal to 1 8th which is the same as probability of A times probability of B times probability of C.

So, it goes little bit. So, this does not imply this, this crucial thing we wanted. Whereas in A, B, C are independent then the probability multiplies out. So, in some sense we cannot do this pairwise independence is the same as independence. Be very careful, it is a very important aspect of probability to keep in mind. So, that leads us with definition of independence for many events, which I will explain now. So, this is crucial idea. So, what is it? So, now let us do a definition.

(Refer Slide Time: 15:38)



So, let us do definition 1.4.5 this is called Mutual Independence. So, let me write it down properly. So, a finite collection, so let N be greater than equal to 1. A finite collection of events A1, A2 all the way up to An are mutually independent, if I look at the probability of the intersection of events is same as the probability of product. So, here I have to put many such cases. So, let me put in a little bit of in green let us say.

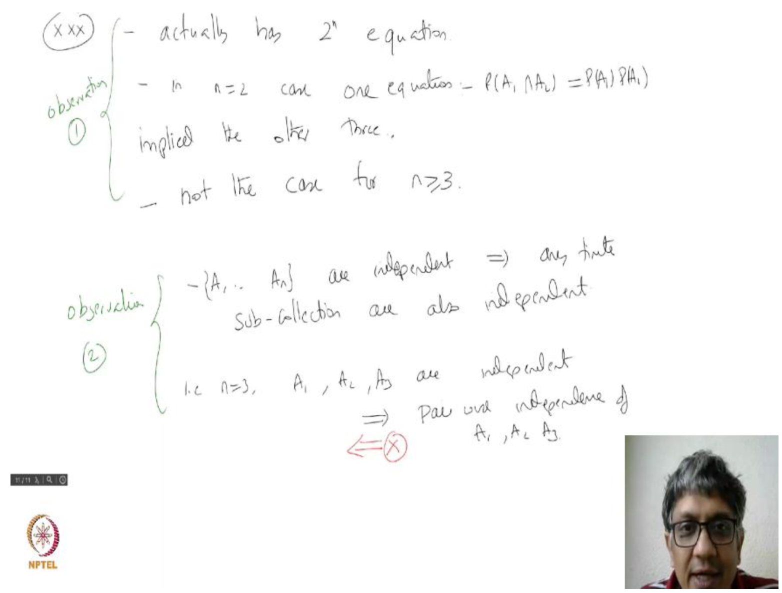
So, probability of Ei is same as probability of Ei where I have to go through all possible combinations. where Ei are either Ai or Ai complement. So is the definition clear? So, the idea is that you want n eventually independent, you have to check this equation for all possibilities of Ai being Ai complement, all with Ai being Ei being Ai, Ei being Ai complement or Ai being Ei.

So, essentially that you have to check 2 to the n equations to check independence and then you can say an arbitrary collection of events, arbitrary collection of events At with t in some index set

I, for some index set I are independent. So, here what you do is, here you say if for any finite sub-collection, they are mutually independent. If for any finite sub-collection, they are independent. Maybe if I want to put the correct English I should say if any finite sub-collection of them are independent.

So, it is a crucial sort of idea independence means you must, you must check these let us call these as triple star. So, what are the observations? So, the observations are the following, so triple star includes 2 to the n equations not one equation. Unlike the n equal to two case, we did not have to specify all four equations, one equation implied all four. For n bigger than equal to 3 it is not the case as we observed.

(Refer Slide Time: 19:52)



ie $\dfrac{P(A \cap B)}{P(B)} = P(A) \iff P(A \cap B) = P(A)\,P(B)$

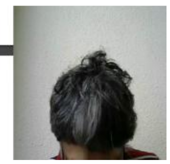
(if $P(B) > 0$)

Definition 1.4.2 (Independence) :- Two events A and B are independent if

$$P(A \cap B) = P(A)\,P(B). \quad - (**)$$

(Seen that if $P(B) > 0$, then $P(A|B) = P(A)$ $-(1)$

$P(A) > 0$ then $P(B|A) = P(B)$ $-(2)$)



NPTEL

Suppose

A_1, A_2, A_3 – [Quantity independence]

(**) – its tempting to say they are independent

if $P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$

– [Caution :- may not imply A_1, A_2, A_3^c are independent unlike in (**)]

– its tempting A_1, A_2, A_3 are independent if

A_1, A_2 are independent

A_1, A_3 are independent

A_2, A_3 are independent

[Pairwise independence]

$\Downarrow$ ⊗

independence

NPTEL

$B = \{\text{Second toss is a Head}\}$

$C = \{hhh, hht, thh, tht\}$

· Seen earlier A & B are independent.

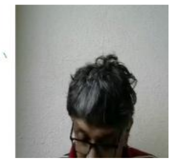
$P(A \cap C) = \frac{2}{8} = \frac{1}{4}$, $P(C) = P(A) = P(B) = \frac{1}{2}$

$P(B \cap C) = \frac{2}{8} = \frac{1}{4} \Rightarrow \boxed{\begin{array}{l} P(A \cap C) = P(A)P(C) \\ P(B \cap C) = P(B)P(C) \end{array}}$

$\therefore$ A, C are independent & B, C are independent

— A, B, C are pairwise independent.

But: $A \cap B \cap C = \{hhh, hht\}$



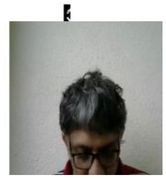
Definition 1.4.5 [Mutual Independence] let $n \geq 1$.

A finite collection of events $A_1, A_2, \ldots, A_n$ are mutually independent if

$$P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i) \quad \text{—} (***)$$

where E_i are either A_i or A_i^c.

An arbitrary collection of events $\{A_t\}_{t \in I}$, for some index set I, are independent if any finite sub-collection of them are independent.



So let us just write the observations down, so one is that triple star actually has 2 to the n equations. Maybe I should write that in green, so the observations are clear. Actually as 2 to the n equations, in n equal to 2 case, 1 equation probability A, A1 and A2 equal to probability of A1 times probability of A2 implied the other three. That is, so that, that is one equation, let me write it in English, so it is clear one equation that is double star implied the other three. But this as we saw in the example this is not the case in, not the case or n beginning, this is observation 1,

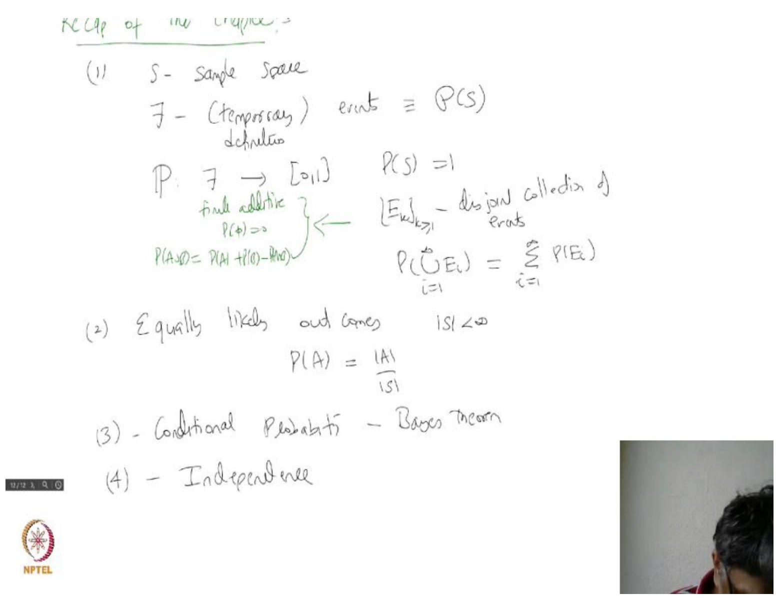
Observation 2 is the following. So A1, A2, An are independent, this is actually implying any finite sub-collection also independent. So, that is in the n equal to 3 case A1, A2, A3 are independent will actually imply pairwise independence. But we already seen the reverse is not true, diverge independence does not imply this. So this is one of the key sort of ideas of this, this phenomenon that you will be careful when this happens. So, in the n equal to 2 case you get lucky, let us just recap quickly, we will stop for the day.

In the n equal to 2 case you get lucky because this one equation implies all four equations. In the n equal to 3 case itself if not one equation you need many more things. Just probability of A1 and A2 and A3 splitting up as a product may not be good enough. So, what you have to do is you saw an example where pairwise independence does not imply actual independence. So, what we have to do is we have to go over all possible complements and events in the collection and you have to check all equations are true. Only then you say the, the collection events are independent.

And the, the two observations are that to check independence for finite-sub collection you will actually need 2 to the n equations, and the advantage of this is that in one shot these 2 to n equations will also give you any sub-collection independence. So, one implication is obvious but the reverse implication is not true.

So, even here this is not true. This you have to be careful that going backwards is not going to be the correct thing, going forwards is okay. Independence is a very crucial property we will see many applications as we go on. So, this finishes the sort of the basic concepts in probability. In terms of. So, let me just quickly recap what all you have done in this chapter.

(Refer Slide Time: 25:05)



So, just quickly recap and then close for the day. So just a recap of this chapter. So, one feature we did was, we defined what a sample space was, we defined we had an event, collection of events, we had temporary definitions so far, which we would take, we will pick (()) (25:34) for some time. Definition of events, this is just the as of now for us it is just the power set of S, all subsets of S.

We defined P to be a probability a function from f to 0 1. It satisfied two things a probability of S was 1 and if Ek are disjoint collection of events, then probability of the union of i equal to 1 to infinity, i is the same as the sum from i equal to 1 to infinity of probability (()) (26:19). That is one thing we start off with. The second thing we discussed was, that was this idea of equally likely outcomes. So, that was the basis of our understanding and analysis after that and we have a finite set, probability of s was finite and the chance of any event A was the size of A by the size of S.

The third concept we discussed was a conditional probability and this include a key theorem called the Bayes Theorem. And the fourth one was the notion of independence. And in between all this we strung together these ideas of, from this we had these other ideas that, probability was finitely additive, that is across finite unions also same thing happens, probability of the empty set was 0 and we also had the fact that the probability of A union B was probability of A plus probability of B minus probability of A. These are all the things that we have done in this chapter. So, now we will use this in the next chapter to sort of understand repeated trials on the same experiment and so on.