Introduction to Probability - With Examples Using R
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Independence - Part 01

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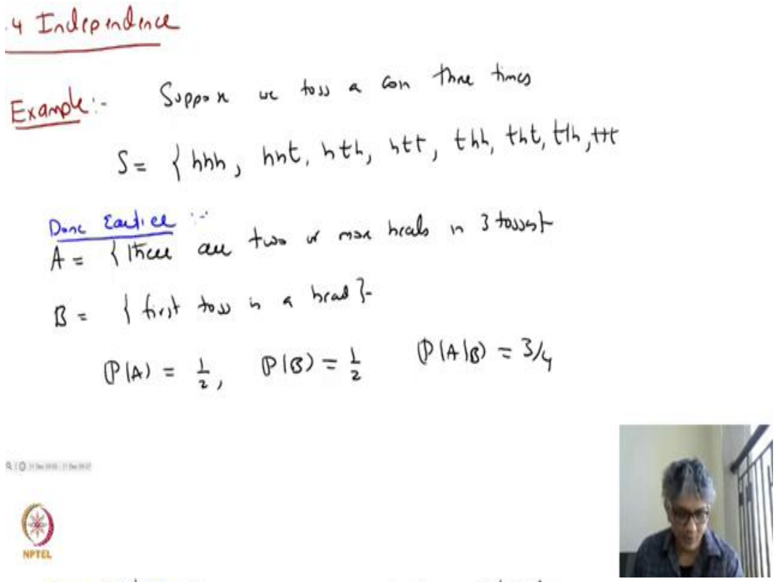

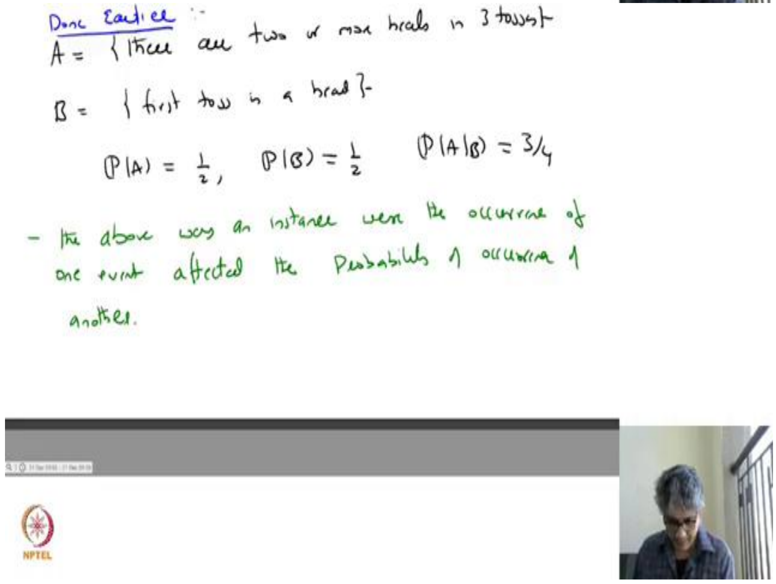

Okay, so, now, I will do an important concept called Independence. So, very, it is a very important concept and it is so important that. And people assume it is easy and make mistakes on it. I do not know, if I said that correctly but, so it is called Independence. It is a very important concept.

So, we saw the example before, let me write the example down again. So, the example is the following, suppose, we toss a coin three times, let us go back to our old coin toss example, then we know that the sample space S is going to be equal to head head head, head head tail, head tail head, head tail tail, and then tail head head, tail head tail, tail tail head, and then tail tail tail, that is a sample space for three tosses of a coin.

And I defined this idea that, the example I did before was I said, let us say let me call it event A, as event A was there are two or more heads in three tosses, was 1 event and B was the event that the first toss is a head. And we had checked that the probability of A was a half, probability of B also was a half, but the probability of A given B was three quarters. So, here are our two events, where of A and B the occurrence of B affected the occurrence of A minus, and this was our foundational idea of understanding conditional probability.

Now, I want to sort of discuss events, where occurrence of 1, does not affect the occurrence of the other. So, now, let me do another example. Now, so, the above was an instance where the occurrence of one event affected the probability of occurrence of another. So, in this case, that the occurrence of B enhance the occurrence of A.

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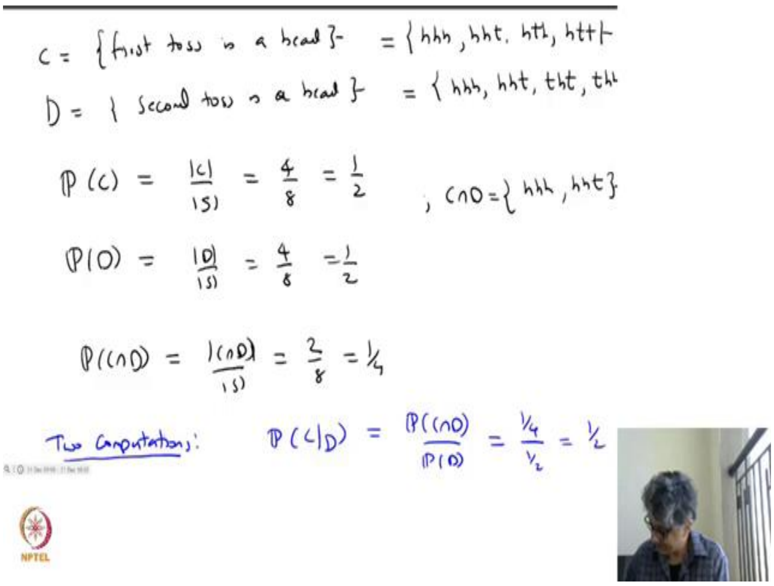

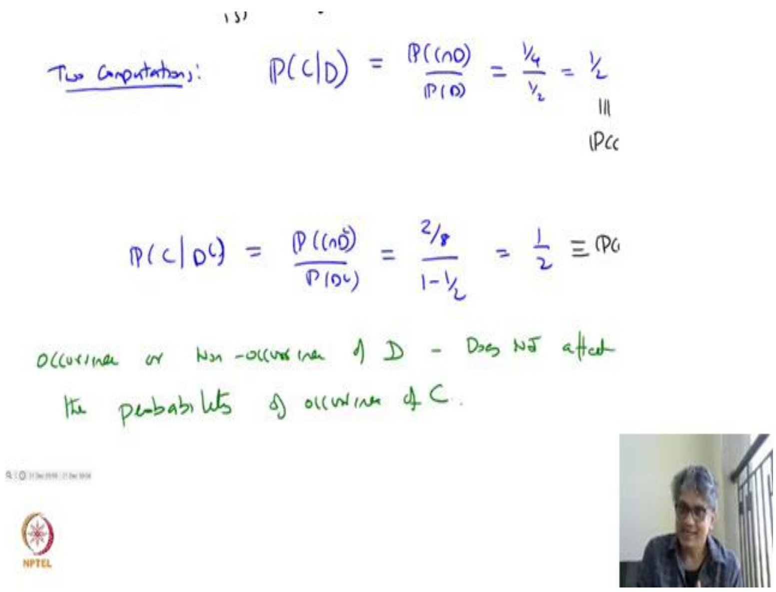

Now, I want to do the following. I want to take a different event, let us call it event C. And let us say, the first toss is a head, and let us say the event D is the second toss is a head. So, now clearly, that A is just the first toss of the head is just that hht, again hht, then you just have hth and then htt, that is the first, that is the event that first of all, that is what C is. Second, toss of the head is just allow everybody hhh, so the second toss will be fixed at hht, and then just go that is it right and then one more thing, there is one more, this thh, very good.

Similarly, again, here probability of C is size of C divided by size of S, which is again 4 by 8, which is one half. And, sub probability of B is again the same way, the size of D divided by size of S is again 4 by 8, it is again 1 by 2. Now, what is C and D now? C and D is the event, that the first and second was a head that means you have hhh, that is two elements there, and then you have hht, and that is it. And, then so you know probability of C and D is again the size of C and D divided by size of S, that is this going to be 2 by 8 and that is 1 by 4.

So, now, if you go and do the chance of, you do two computations, let us do two computations. So, probability of C given D, that is the same as probability of C and D divided by probability of B that turns out to be a quarter divided by probability of D is a half and that is going to be again a half. So, let me get this properly. So, what we have shown is, probability of C given D is also a half and that is the same as probability of C, that is what we should get before.

So, the occurrence of D, does not seem to affect the occurrence of C at all. Similarly, let us do the following, let us do simpler one, let us do something like C and D complement is what now, D complement is that the second toss is not a head, that is again, that means C is the first toss of the head, the second toss is not a head that means, these two guys. So, it is just again going to be hth and htt, that is what C and D complemented is. So, you can also see the probability of C, given D complement is also equal to probability of C and D divided by probability of D complement, C and D complement of our probability complement and probability of C and D complement is 2 by 8 and the probability of D complement is again one half because probability of D is one half, so, one half 1 minus half.

And this also turns out to be one half, it is also same as probability of C. So, what we observing here is that, the occurrence or non-occurrence of D, that is D or D complement occurs, does not affect the probability of C, the probability of occurrence of C. So, what we are seeing is that, the occurrence of one event does not affect the occurrence of the other. So, in some ways that, no matter what happens. So, the probability of C is going to remain the same regardless of D happening again. So, that means C and D are kind of independent chance, which is what we will formalize the next definition.

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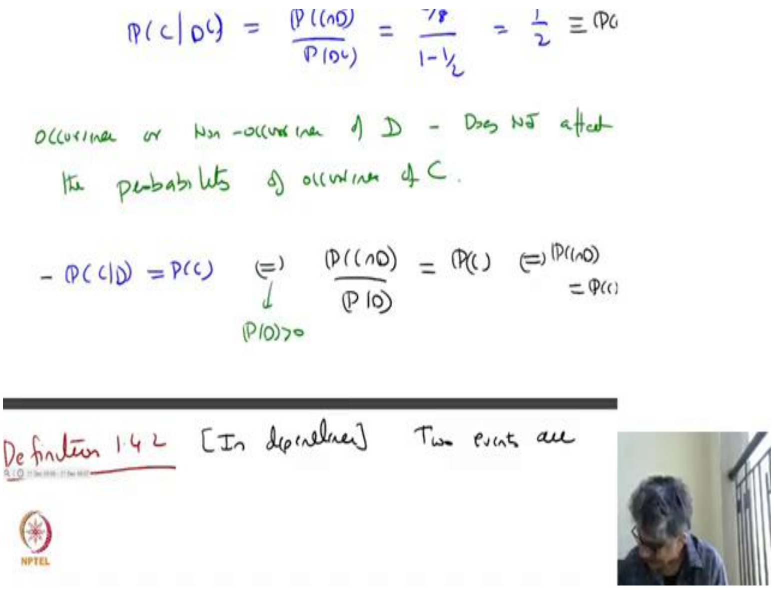

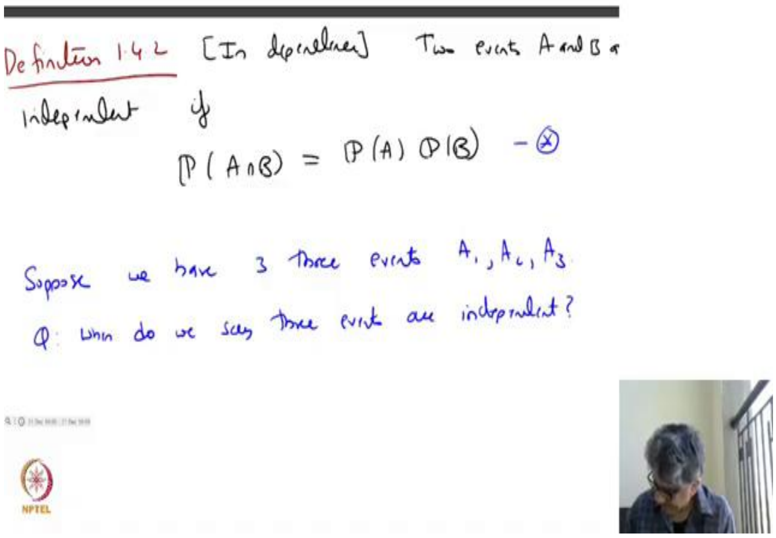

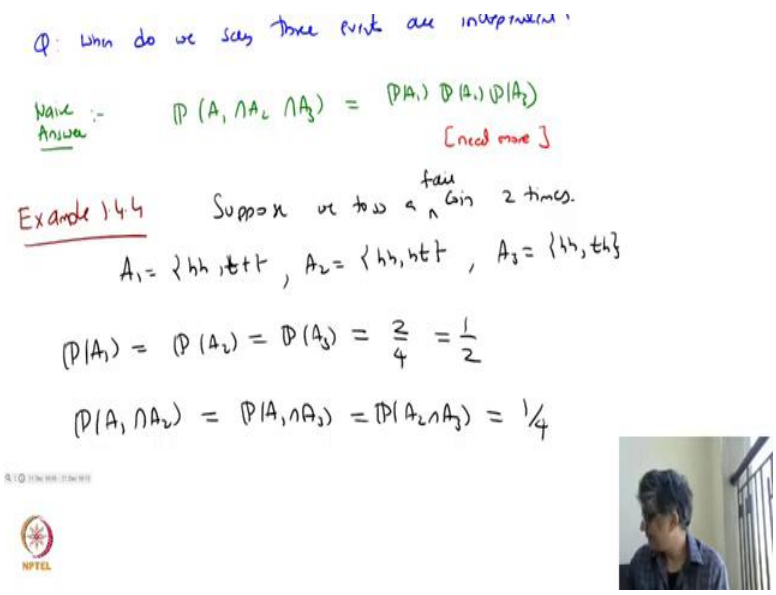

So, here is a is a very important definition of probability, this definition 1.4.2 here, this is called Independence. We say two events are independent of each other or independent. Here, the chance of A and B is chance of A times chance of B. So, this is the no, it is a definition of Independence. So, why the occurrence of C is same as occurrence of C given D and C given equals of D complement is, by definition not to be independent, if the probabilities match.

So, let me just rewrite the definition in terms of what is given above. So, here, I showed you that, so why did I, why is that definition equal to what I wrote before, is that, let me write this in blue, the probability of C given D is equal to probability of C is equivalent to the fact that probability of C and D divided by probability of D is equal to probability of C. And, this step required that probability of D is positive, but if I use my little convention, then I can go back and say the same as probability of C and D is equal to probability of C times probability.

So, of course, here in both these things and require probability is positive. In Independence, I do not require positivity, I just say that if this is true, this is called Independence, this is definition. So, you say two independents, you say independent A probability of A and probability B is equal to probability A N. So, now, from true events A and B. So, now, we have seen examples of independent events and examples of events, who are not independent.

In the previous example, so, we saw that the previous example C and D were independent, but A and B the example were not different. Let me just recap that. So, here I knew that independence is that A and B independent. Here, I showed you that C and D were independent but A and B were all independent. Let us probably A and B was not equal to probability. So, this is a name, so, you think of two events independent, if the conditional probabilities are the same as the original problem, that is why you call them independent, very nice.

Now, let us try and understand, I want to dwell on these equations a little bit, let us try and let me try and do that. So, two events are independent, if probability of A and B is same as probability of A, that is positive. Suppose, we want to extend this to three events, let us call this a star. Suppose, we wish to extend it to, suppose, we have three events A1, A2, A3, and we will not understand, what it means to say that 3 events are independent.

So, question is, when do we say three events are independent. So, one way to think about it is that, you can just say that the formula star holds. So, one naive way for following, let us do first naive one, you can just say, okay, I just take it as probability of A1 and A2 and A3, but by definition of the above is same as probability of A1 times probability of A2 times probability A3. So, now, this is an issue, so, now, if can I just say this to say that three events are independent.

So, here comes the, here, comes the clash with an intuitive notion of what we think is independent, independent means that kind of the conditional probabilities give the original probability, that is why we call the independence. But is this is enough, but we definitely want this, is this is enough. So, let us do a simple example to see that this may not be enough. You need more, so, you need more to assure three events are independent. So, let us try and see simple example.

So, let us do the example 1.4.4. So, suppose, we toss a coin two times, coin two times. So, you toss the coin once and toss the coin again. And, let us say toss it fair. So, the probability of heads is half the probability of tails. So, then you would say that A1 is equal to head head and this tail tail, and A2, let us say the event that you get up head in the first toss and A3 is the event, let us say you get a head head and then tail head in the second toss. So, A1 is the event, you get head head or tt, there is two heads or two tails.

A3 is the event you get two head, one head in the first toss, the other one in second toss, A3 is the fact that you are getting head in the second toss, then one can easily see, that probability of A1 is equal to probability of A2 equal to probability of A3. So, there are four possibilities in the sample space and you have two of them, so, it is 2 by 4 and that is just the half. So, they are all the same. Then probability of A1 and A2 is what now? And probability A1 and A3 they are all the same because you look at it and A2 and A3. So, A1 and A2 has only one element in common, A2 and A3 again have one element in common, A1 and A3 again have one element in common, they all have only 8 H in common So, it is going to be 1 by 4, is that clear?

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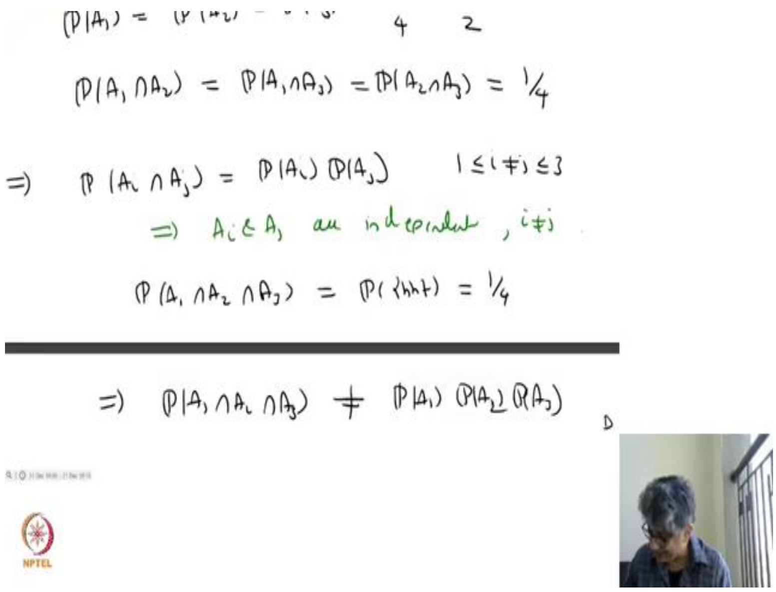

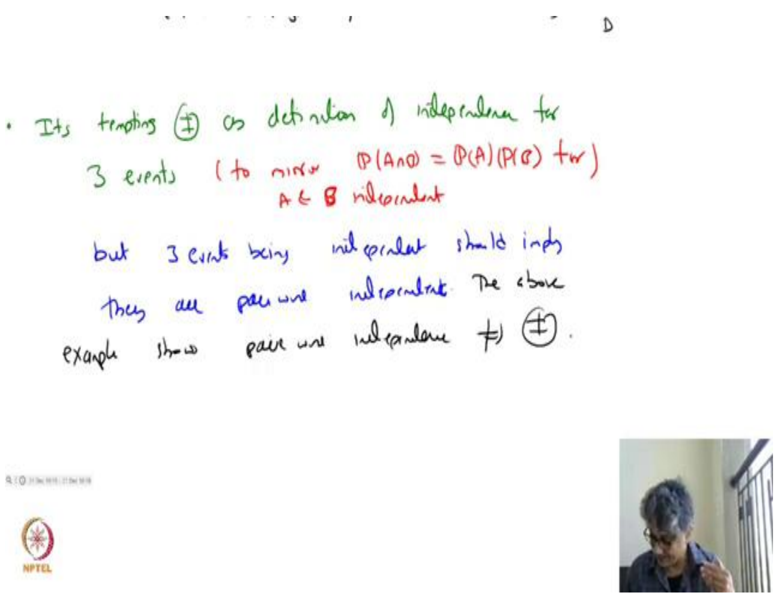

So, from the above, this clearly implies, that probability of AI intersection AJ is the same as probability of AI times probability of AJ for 1 less than equal to I not equal to J less than equal to 3, that is just straight forward, you just multiply the two things, half and half is a quarter and all the intersection are clear. This clearly implies, so, we are in business now, you can clearly say, now, that A1 and A2 are independent.

Well, let me just go a little bit before, all the pairs are independent. That is AI and AJI, and then for I not equal to J. Now, comes the crucial part. So, I wanted to understand, what more do I need right. So, one idea could be that I have probability of A1 and A2 and A3 is equal to this and all the pairs are independent, that is one idea, but that may not be the case because you look at this guy, but you look at probability of A1 and A2 and A3.

So, what is the combination of all three? It is just head head, so, it is just probability of the event head head tail and that is just one quarter. So, probability of A1 and A2 and A3, for the probability of head head is 1 point. So, this implies that probability of A1 and A2 and A3 is not equal to probability of A1 times probability of A2 time probability of A3. So, in some sense if you have three events and they are all pairwise independent, that is not necessary to guarantee this 1 by 4.

So, you cannot have this. So, going back to our investigation, what is going on? So, if you want to know three events independent, it may not be, one thing is that if you just assume this line it may not be enough because you know this does not guarantee, the fact that the after A2 and A3 are, do not affect that with A1 and solve the 4. So, you do not know that, so, you need more. One way to do more is you say, okay fine I go and say all parallel independence but that is not enough because parallel independent does not imply this equal, which you would like to have.

So, here is the summarizing remarks. So, a, so, it is tempting to do the following, that let me call this as dagger. It is tempting to have dagger, as definition of independence for three events this because it is to mirror the probability of A and B is equal to probability of A N, transformative B for A and B independent but (())(22:33) but see if you have two events independent then they must be famous independent. So, but three independent, three events being independent, three events being independent, should imply they are pairwise independent.

But the above examples shows, the reverse is not true. The above example, shows pairwise independence does not imply that, that is the first factor. So, pairwise independence is not is not sort of a global concept to assume three events independent and three independents should imply pair wise independence, that is something we understand. So, we need a, we need to be careful about what we mean by three events being independent. So, here is the fundamental observation, see I know, if you realize this I just wrote down star, for probability of A and B equal to probability of A that of B as being two individuals.

And I, but when I did the example, when I did the example for you, I said that probability of C given D is half, probability C given the complement also is half and they were both probability of C, that is when I concluded that C and D are independent, that is the occurrence of C and that is does not depend on the occurrence of B or D complement, but then in the definition I somehow wrote only one equation, probability of A and B is equal to probability that positive B.

Somehow, I was a little glimpse, I just did then clarify to you why one equation was enough for independence. But here is the mathematical sort of justification for this, why this is enough, star actually is these four equations, let me write that down. So. what is it that we need more?

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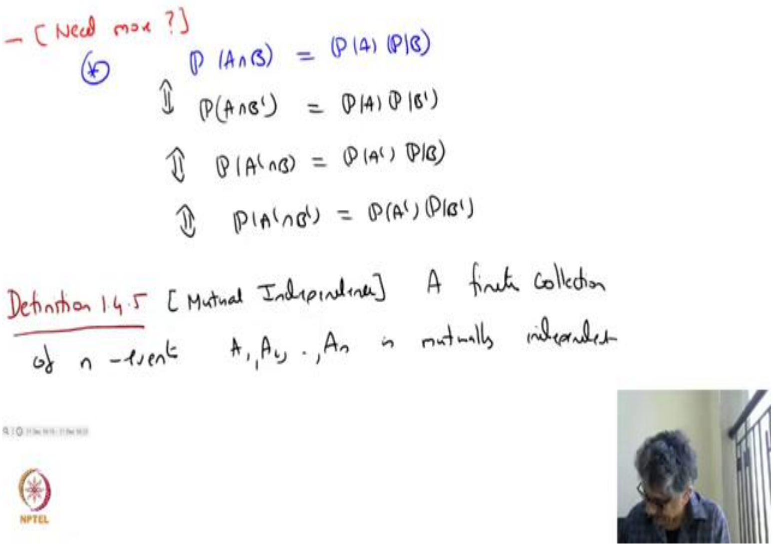

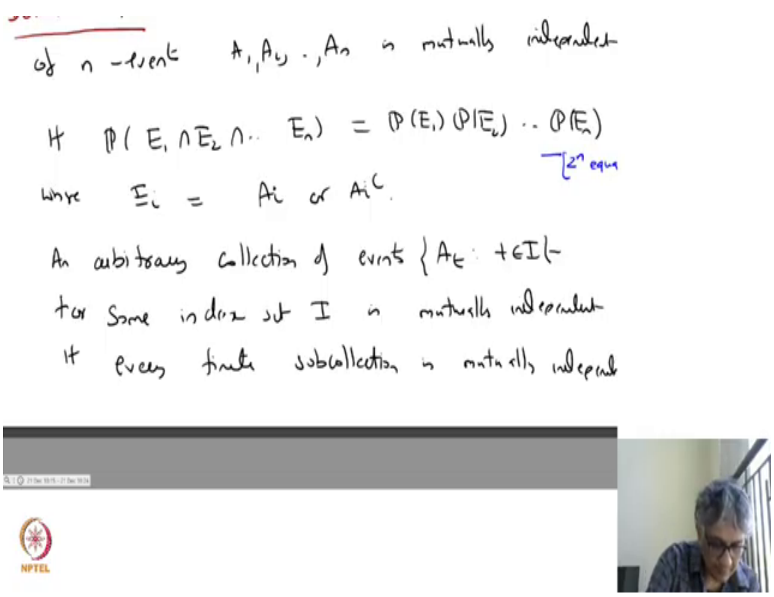

So, star with this probability of A and B is equal to probability of A and probability of B is actually four events because from this you can get, you can just check, this will actually imply and be implied by probability of A and B complement, is probability of A times probability of B complement is actually also equal to probability of A complement and B equal to probability of A complement times probability of B and that is equivalent to probability of, what is the last one? A complement and B complement is equal to probability of A complement and probability of B complement.

That means A and B are independent, if compliments are independent or A and B complement independent or A compliment at the end. So, that one equation gives you four equations, but the three case you do not do that, that only equation will give you all. So, that is the definition you will need for these two independents. Let me write down and end the class, definition 1.4.5 this is called mutual independence.

So, it is a very important definition, just keep it in mind. A collection of N events, A finite collection of N events, say A1, A2, up to AN is mutually independent, independent if probability of E1 and E2 and E3, sorry A1, Probability A1 and A2, wait a minute, sorry, So, I want probability of E1 intersection E2 Intersection EN is equal to probability of E1 into probability of E2 into probability of EN, where EI are either AI or AI complement, that means, this is like a collection of two event equations, that must be true for independence.

So, this is like a collection of 2 to the N events, so for the three case, you will need two and three equations and eight equations to guarantee independence of A1 A2 A3, you want all the complements and them to also involved. Now, we can also generalize a little bit, an arbitrary collection of events AT, T is up index at I, so for some index set I, I is mutually independent, so, (())(29:06) independent, if every finite sub connection is mutually independent, so let me just quickly recap independence for you.

So, we began with the following idea, that we had an example of tossing three coins, we knew before that if you have A is the event, there are three or two or more head in three tosses and B was there when the first toss is head, we knew the occurrence of B affected the occurrence of A, then its probability of A given B was not the same as probability of A.

But then I said okay fine, let us do something else, let us look at, let us use the kind of independent functionality that is the first toss does not affect the second toss in some sense intuitively. So, I defined two events C and D the C was the event, the first toss is head, D was the event the second toss is a head, then I found out the chance of C was half the chance of D was half and the probability of C given D was a half as well. So, was the probability of C given B complement also half, they both match probability of C very, that means the occurrence of C nor the occurrence of D or the occurrence of C occurrence or non-occurrence of D does not affect the probability of occurrence of C.

So, that is what we think of independence between two events C and D, if the probability of C given D is same as probability of C given D complement, the same as probability of C we will say that C and D are independent, that is what we. So, then I quickly turned around and said okay fine what you said here, define notion of independent true events. So, I said two events are independent, if probability of A and B was probability of A times probability of B.

So, I use star as definition, because I do not have to worry about probability of B being positive or zero, because if I use conditional probability definition, then I will have to understand that probability of D is positive by the definition, but I have used this convention that probability of C and D is 0 (())(31:59) or probability 0, then I can use this slide right here. So, that is why I use the convention that 2 is independent if probability A and B is same as probability A and B.

And, if probability B is positive and probability of A is positive, then I immediately know that probability A given B is same probability of A and so on. So, if the probability B is 0 automatically, I assume that A and B are independent, no matter what A is. So, now, that I tried the fact that, if I have three events what do I do, how do I define events. So, one way is, tempting is from star, is to write down as probability A1, A2, and A3 is same as A1 to AN, sort of an intuitive way of, sort of generalizing the fact that three events again, but okay let us do a check, we said, we might need more and why we need more is that okay?

Three independent we expect them to be mutually independent as well pairwise, but then are they equivalent, so that we immediately check that I give an example of two events, three events A1, A2, and A3 which were, there was independent that is P of AI, N AJ where P of A, but P of A1 A2 A3 was not the same as the product of A1 probability of A2, that means dagger did not hold. So, that was independence, does not mean the events are Mutually Independent.

And the observation for that, was that the star out here, was one equation. But actually, it had information about four equations because they are equal, you can try and show the simple exercise to show earlier, to show that the probability of A and B is equal to probability of A and probability of B times probability B, implies the other three equations and vice versa. So, that lot of definition mutually Independence, we say two events are mutually independent or any of its independent, if 2 to the N equation holds, that is probability of E1, E2, and EN is equal to probability E1, where E1 are both EI and AI complement.

And then we say an arbitrary mutual independent, if any finite separation, so, it is kind of a very important idea to understand that if AI are independent, means AI and any budge, any combination of them then the complements also independent, and that is guaranteed by the two given equations, and not by just one equation that is because in the N equal to two case. We got one equation down but that contains four equations.