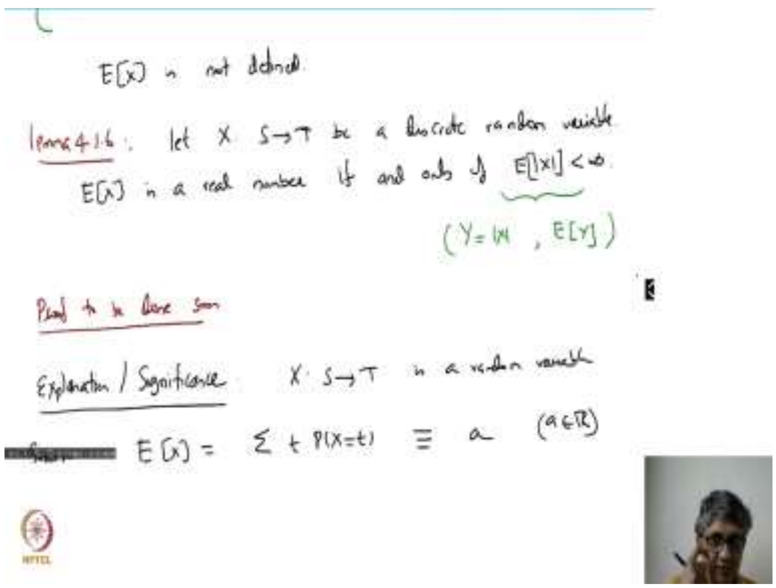
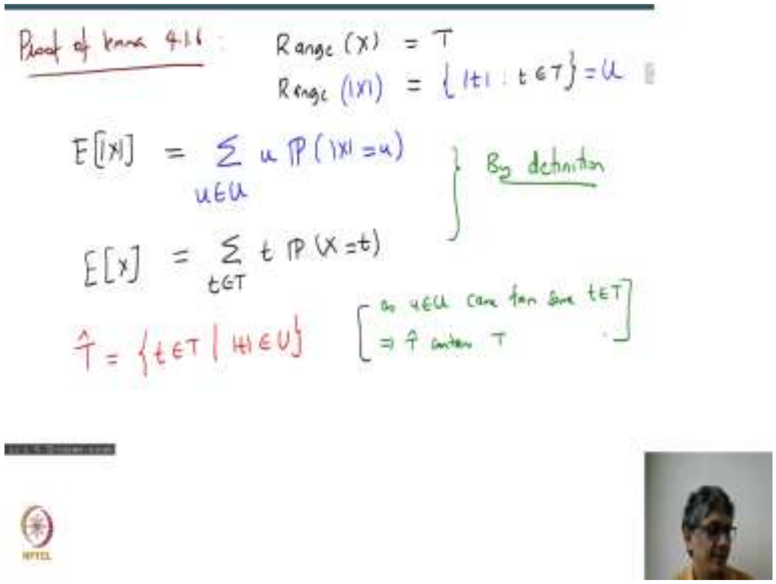
Introduction to Probability with Examples Using R
Professor. Siva Athreya
Theoretical Statistics and Mathematics Division
Indian Statistical Institute, Bangalore
Lecture No. 22
Properties of Expectation

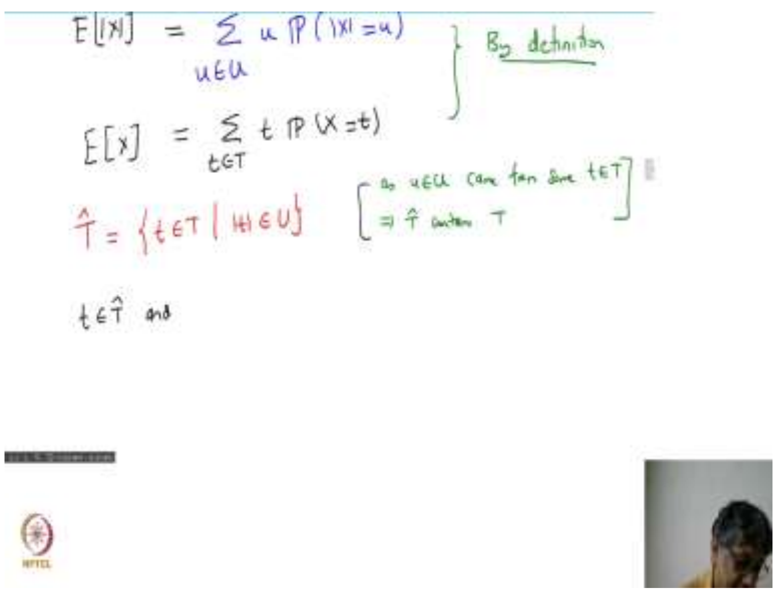
(Refer Slide Time: 00:16)



So, now we here so let us something think about what does E of X tell you? We will try and little later we will be as I want to some algebraic compositions today so to finish off the technicalities, we will come and discuss the intuition again E of X (())(00:31). So, today’s class I want to start about this, so I wanted to just show that lemma, I want to show the proof of this lemma that is E of X is finite if and only if E of mod X finite. It is called the important result and it is very powerful, so and it is kind of certain to understand but the proof is cannot be eliminated.

(Refer Slide Time: 01:06)





$$E[|X|] = \sum_{u \in U} u \, \mathbb{P}(|X| = u)$$

$$E[X] = \sum_{t \in T} t \, \mathbb{P}(X = t)$$

} By definition

$$\hat{T} = \{ t \in T \mid |t| \in U \}$$

[as $u \in U$ can form some $t \in T$
$\Rightarrow \hat{T}$ contains T]

$t \in \hat{T}$ and $t \notin T$ $\Rightarrow$ t is not in Range(X)

$$\Rightarrow \mathbb{P}(X = t) = 0$$

$$E[X] = \sum_{t \in \hat{T}} t \, \mathbb{P}(X$$

$$\Rightarrow \mathbb{P}(X = t) = 0$$

$$E[X] = \sum_{t \in \hat{T}} t \, \mathbb{P}(X = t)$$

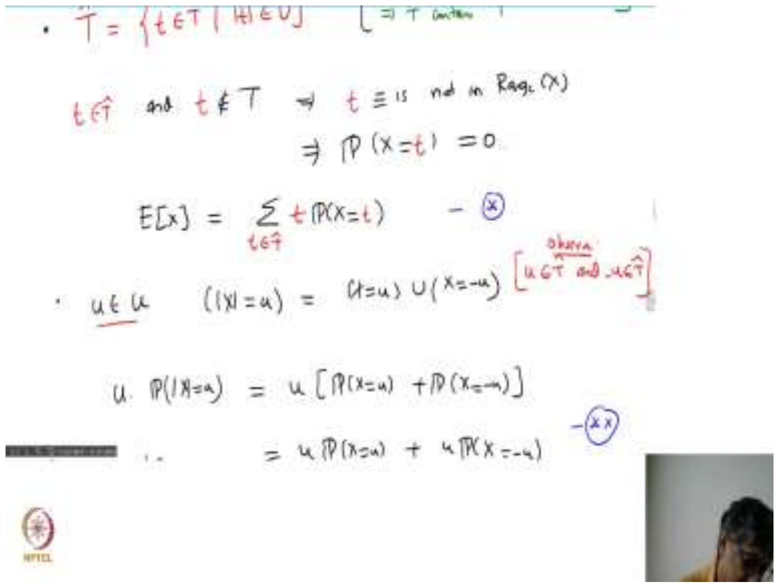
· $u \in U$ $\quad (|X| = u) = (X = u) \cup (X = -u)$ $\quad$ [$u \in \hat{T}$ and $-u \in \hat{T}$ shown above]

$$u \cdot \mathbb{P}(|X| = u) = u \left[\mathbb{P}(X = u) + \mathbb{P}(X = -u) \right]$$

$$= u \, \mathbb{P}(X = u) + u \, \mathbb{P}(X = -u) \quad -(**)$$

[True even if $u = 0$]

$$= |u| \, \mathbb{P}(X = u) + |-u| \, \mathbb{P}(X = -u)$$



So, proof of lemma what I call is 4.1.6, so proof of lemma 4.1.6. So, now what is been, so now we have to show E of X is finite if and only if E of mod X is finite, let us say range of X is T, so what happens then I defined I refer the range of mod X, so range of mod X think about it, what will it be, you have to little bit careful let me erase this thing and write them blue range of mod X could be what now, so we will have to take everybody in T, but takes its absolute value, so its mod t such that t was T. And that is the range of mod X.

So, now you do E of mod X we can go and find the distribution on mod X and do the whole thing, but if we do not know mean as if now we do not know nothing about X we just know X random variable, so let E of mod X would be the same as the sum from t, so let me call this as let me erase this lemma, so let me call this set as u as a range of X, range of mod X, so let me same as the sum over u and U number u and to the probability mod X is equal to u.

So, now it is a simple exercise to just conclude that this is the same as or before I made a conclude now that times see E of X is defined as sum t and T number t (())(03:08) probability X equal to t, so we would have to sort of connect E of X with E of mod X, so we have to say E of mod X is finite and E of X is finite and vice versa. That is the thing, this is our definition.

So, now let I had to connect that two sums, so the trick is the following, so you have to do the reverse way, so here we will (())(03:47) u to t (())(03:48) well let me define the little bit differently, so let me call it T hat as the set of all t E T such that mod t is in, so I am going to (())(04:04) and I am not going from t to u but I am going from back from u to t. So, now since every u came from some t, T hat contains every element. So, as (())(04:23) as u in U came from something t this will imply that T hat contains all other T, that is one observation can make, so T hat contains T.

Now the next observation is the following. So, for let us t is in T hat and I am writing in red other by it is clear for you, t is in T hat and t is not in T, then this will imply that t is not in the range of X, this would imply what do you mean by t is not in the range of X for random variable it is crucial thing is a reply that probability X equal to t would have to be 0. Because X distributes all these probabilities on it range. So, anybody outside this range X probability 0.

So, therefore what you could do is even though T hat in that largest set you could extend you could think of E of X as it is okay because probability equal 0, so I could just say this is a sum over t and T hat, t times probability X equal (())(06:11) so this is a two observation we can make, so let me re-write this in red so this t is always in red, so this t is in red, so this is in T and not in T hat, now t is in T hat not in T this is simple pattern.

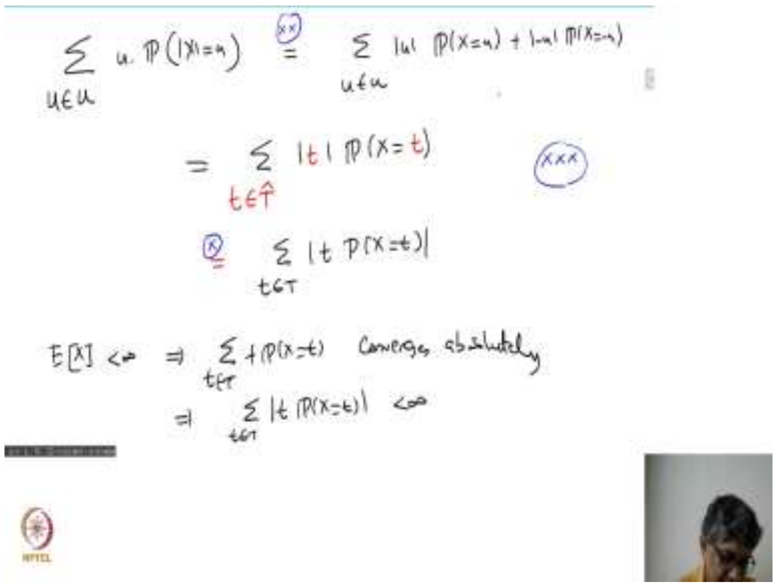
So, E of X can be extended to T hat in this session, because there are so zeros so this could, so now you go reverse with, so now you say this is one competition variable, the second one is you take a u and u take a u in U, you look at the event mod X is equal to u, look at the event mod X equal to u. What is that going to be? That is going to be the event X equal to u or X equal to minus u that is even mod X.

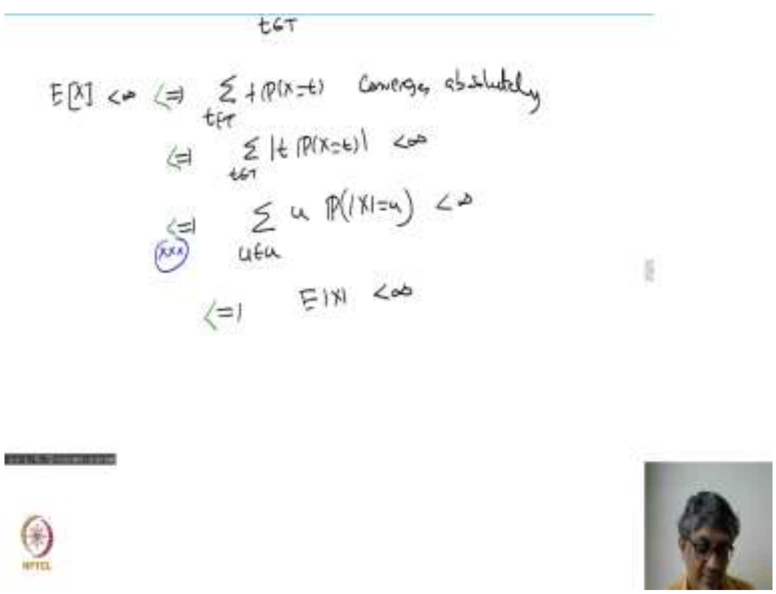
So, then and you know due to the following fact u is in T hat and minus u is also in T hat, (())(07:40) because u is in u and you know that using T hat and minus u T hat, observe we have this, so therefore u times probability mod X equal to u that is the (()(08:09) value that is the same as u times probability X equal to u plus u times probability X equal to minus u. So, it will just a (())(08:23).

So, now this is the same as this is the minus u here, I just put (())(08:51) doing start again, so u this so same as u times the probability mod X equal to u is probability X equal to u plus probability X equal to minus u because you have two disjoint events and that is the same as u times probability X equals to u plus u times probability X equal to minus u and that is the same as I will write it in this fashion use new u is equal to mod u, mod X equal to u and use u is same as mod minus u, so probability X equal to minus u, this is one expressions.

And this is true even when u is 0, but I will not said so this I will write a small observation here, this is true even if u is 0, so it is true, it is trivial. So, once you have this you are in business now, then you can relate the two sums you have this observation here let us call this as maybe in blue let us call this double star, I have this recalling here and this one here as a single star, so now we are almost ready to compete proof. How do you complete the proof?

(Refer Slide Time: 10:32)





You write the following, you write the fact that the sum over u in u, u times probability mod X equal to u, from this side we know that is the same as we do this is the double star u and u such that your mod u times probability mod X equal to u plus mod minus u times probability X could be minus u, so this is the star X here not mod X, X equal to u plus mod minus u times probability X equal to minus u and that is the same as the chance of t in T hat so that mod T we will write this in red now that is how you (())(11:26) t and T hat mod t times a probability X equal to t (())(11:37) red again I move to T hat from T hat I know exactly I how to go to t that is the same as the sum on t and T mod t time mod of absolute value of E times probability X equal to t that the same thing.

But, now I know right now I know the fact that this is just if this right hand side is finite and left hand side is finite, so I just says that if E of X finite so if E of X is finite this implies that the summation t and T t times probability X equal to t converges absolutely and that implies that the sum t in T the absolute value of t times probability X equal to t finite and that is why this about triple star let us call this triple star, so drive triple star, so in triple star I used double star here and I used double star here.

So, my triple star that is the same as the sum from u in U times probability mod X equal to u and that means that E of mod X is small. But now the same argument applies in reverse because you know that you can do the same argument to reverse because you know that E of mod X is finite that implies that this guy is finite, this guy is finite previous colour black, but this guy is finite from double star I can go and say this guy is finite. But this guy is same as saying that guy to absolutely and that means this guy is finite. So, there is really this is a two-way inequality and that is how (())(14:07) so let me show the proof in one shots, so we can have a good view, split to again go here.

(Refer Slide Time: 14:26)

So, let me come back here and make this small and go to previous page, let me come here so now the proof is a little tricky even though it is an elementary proof that will be little carefully count things. So, what you do is you look at the range of X called that t, then range of mod X you call that u, now you know that E of mod X is just summation u and u trans probability mod X, you know summation T tends probability X equal to t is same as E of X.

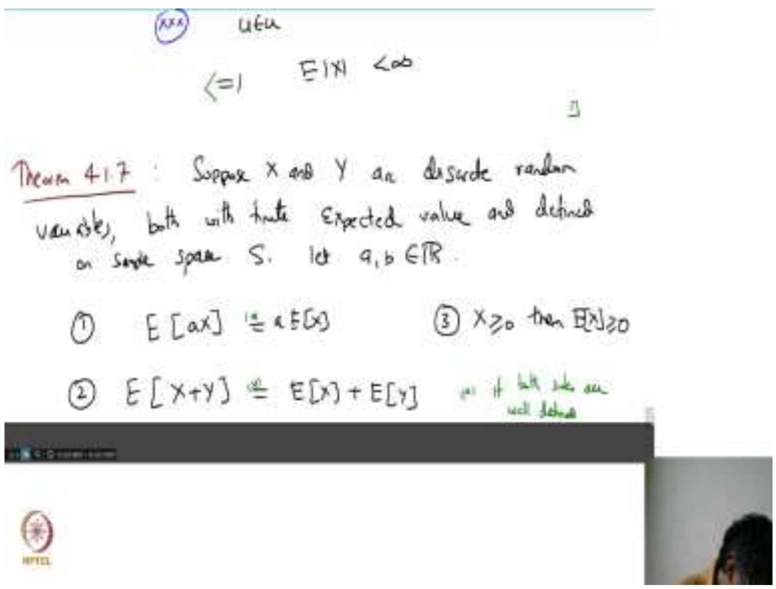
Then you have to connect this range u with T that you know one way, the other way is you take T hat you look at all the elements in T in capital T which are absolute value is in u. Now, if t is in T hat there and T does not belong to t, then so I should maybe I should just I made one small error the way I defined T hat is not correct. So, T hat is all real numbers for which mod T is it, so now if t is on T hat and t is not in T, then t is not in the range of X.

So, chance of X equal to t is 0, so therefore you take E of X is the sum from t in T hat probability X equal to t times T, because what is not in t you get 0 probability and then you gets match up with definition E of X. So, then you just do the reverse you say you observe u in mod X equal to u, splits up as mod u times probability X equal to u plus minus alpha minus u times probability X equals to minus.

So, once you have these two things star and double star from the second figure right hand side, it is kind of obvious now, and you look at E of X it splits up as two parts. So, if the some (())(16:26) absolutely then let me just remove this (())(16:41) so then the idea was that if we see in the right hand side E of X was finite if the series converge absolutely and the converge absolutely was just us looking at the E of mod X, because of this competition that you have done far up and down with T hat and u and t, that gives you the fact that E of X is finite means E of mod X is finite and vice versa.

It is kind of a funky fact and it is important proof, but the proof is a little tricky, even though it is sort of you have to keep your head straight, now let me try to build on this, so I will do one more algebraic factor today and I will stop, I will come back and do the intuitive idea expressions next time.

(Refer Slide Time: 17:59)



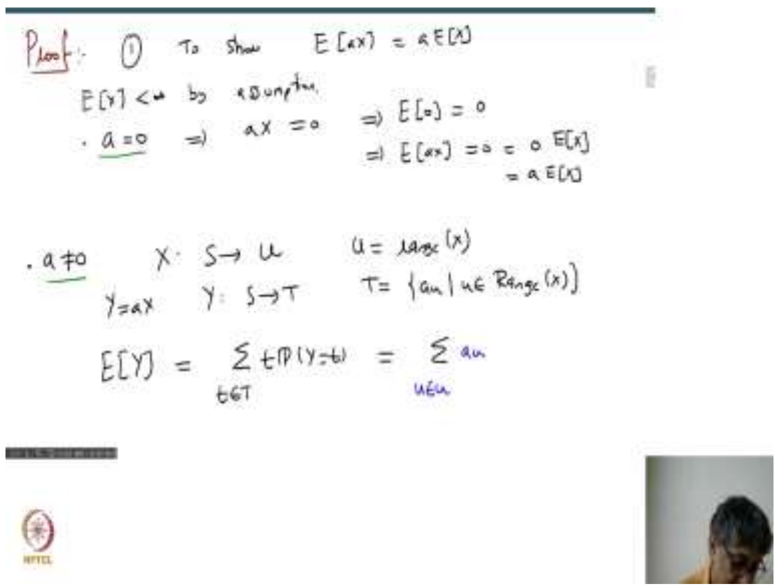
So, here is another theorem that it is also important theorem that we use again again we use by (())(17:56) theorem, so here suppose now I take two random variables, suppose X and Y are two random variables, are discrete random variables both with finite expected value and let us say we define on the same samples this system. So, it is an important fact same sample space S let us say full stop let us say it take two numbers a comma b be two real numbers, then the following facts, first thing is if I take the expect value of the random variable a times x that just gives me a time E X.

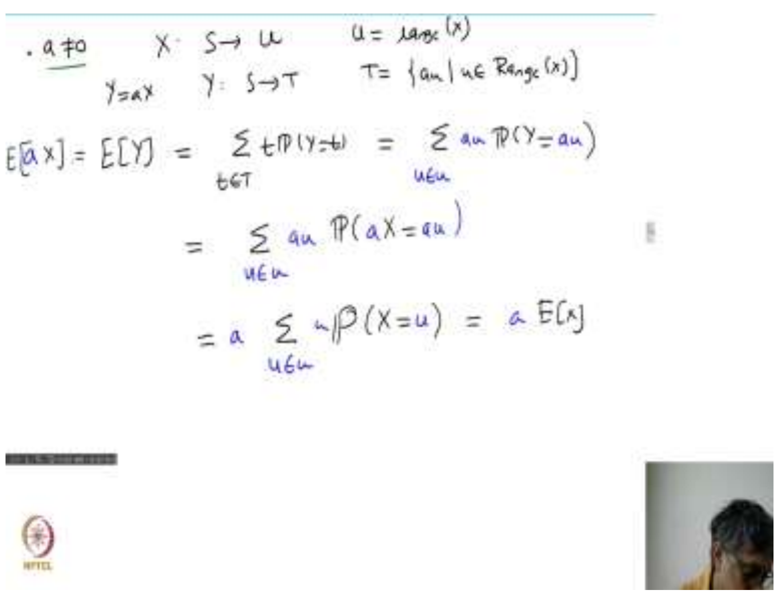
The second thing is E of X plus Y will give me E of X plus E of Y, so the expectation is linear and the third thing is that I have suppose this consequence made on this is that X is non-negative than E of X is. So, now of course now each of these step to be understood in the following way.

This equality, this equality, understood that both sides exist and are finite, only than the equality applies, if one side is not defined you forget about the equality, let us understood that part.

So, in these two cases, let us call them a star, if both sides are well defined, that is when you understand this because it could be possible that because we though it cannot be suppose E of X it could be possible that E of X is finite E of Y is finite then E of X plus Y is finite then all this is true, it could be possible that E of X is minus infinity, E of Y is plus infinity, X plus 1 must define, but this cannot be equality may not fall true. So, the equalities of only happens if both sides are valid. Let me try and proof this theorem.

(Refer Slide Time: 21:25)





So, the first thing I have to show is for proof 1 I had to show a times E of ax is a times E of X, so for that let us say, so to show, so what I have to show? I have to show E of ax is a times E of X. So, maybe I will simplify the theorem a little bit so the way the star maybe not be necessary that means suppose stars here, here I have said that with both finite expected value that means I will just say here into this, I just say both sides automatically well defined, also comes the equality.

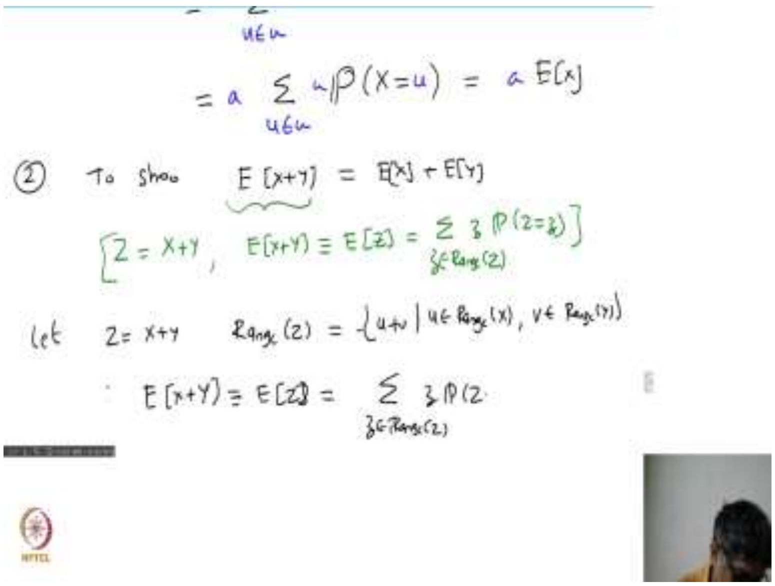
So, E of X, so first thing is I have to observe is that E of X is finite by definition by assumption, so then let us say a is equal to 0 let us say a is equal to 0 firstly if a is equal to 0, a times x is also 0 and this implies the previous lemma that E of 0 is just 0 which implies that E of ax is 0 and 0 the same as 0 times E of X, because E of X is finite, so (())(23:02), so this part is clear.

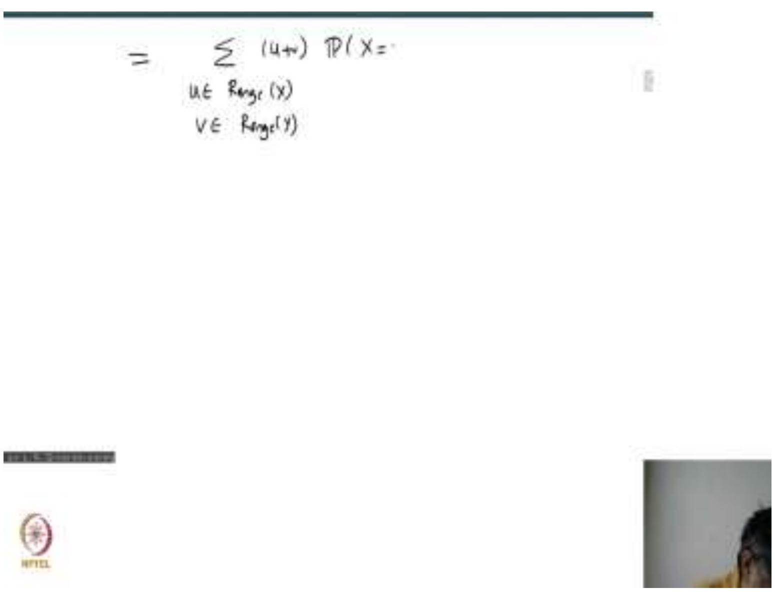
Now, let us say a is not 0, let us do this things, so a is 0 the case is clear, if a is not 0, what happens? So, a is not 0, then X is a function let us say from S to some number let us say u, u is a range of X, what will ax be? So, the random variable let us say Y is equal to a times x, Y is in an S from let us say T, where T is again the set of all a times u such that u is the range of X.

And now it is a simple idea is that simply a Y is what is we going to understand is the same as the sum from u sorry t and T, simple equality t and T, T times probability Y equal to t, is the same as the sum from this can we write this as the sum from let us say u in U, a times u probability Y is equals to a times u, as is by definition of elements in (())(24:34) and then come down then let us say that is the same as the sum over u and u, a times u and this is just the chance that I will write that Y is ax, a times X is equals a times u and now everything is finite so I can add multiply remove, so the same as the chance that a comes out I have the sum, sum is in black again the sum over u and u, u times the probability that here the a just cancels off.

And that is just a times so this equality actually shows that that just a times E of X and if it is observed we start off with just a times X expectation, excuse me, so a is not 0 when you look at Y is equal to ax and just follow to the competition (()(26:11) that is very simple, so that finishes the proof that if Y is E of ax equal to a times E of X.

(Refer Slide Time: 26:27)





The next part is little bit tricky, you will be a little bit careful even though it is again a simple part. So, let us say you have this is a very little bit careful, so now what you do is you want to show in the second part you want to show E of X plus Y is the same as E of X plus E of Y, (())(26:47) so what how do you understand the random the expression on this side? This side is you take z equal to x plus y and then you think of it as E of x plus y is the same as E of Z and is the same as the sum from z to Z there is range of Z, Z time the chance that (())(27:17) that is how you understand the right hand side.

And these two you understand clearly these two, so now somehow you have show that these two are the same, so you have to if you want to show this proof precisely you have to go and compute the distribution of Z somehow once you find it out, you have to get this idea, so let us try and see how to this, it take some amount of work.

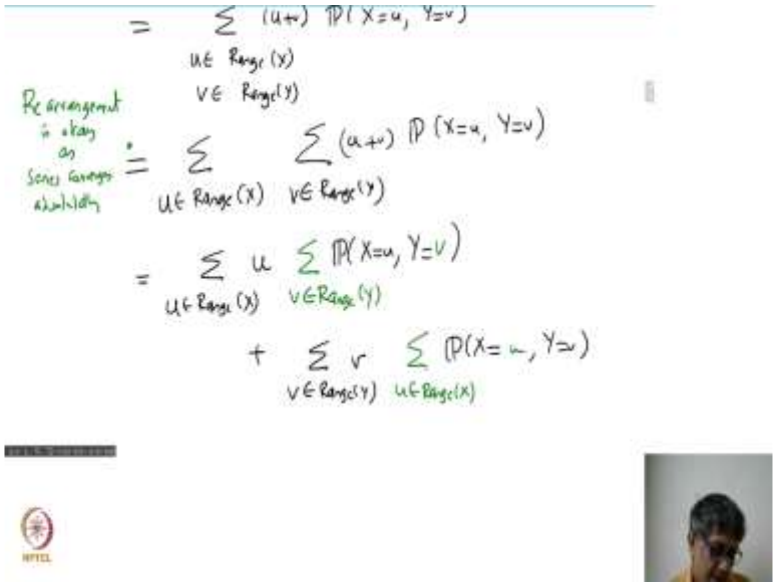
So, the first observation is the following. What is the range of Z? So, let Z equal to X plus Y then the range of Z is the same as let us say S plus T, then S in the range of X and t is the range of T and that is some function equality, the range S splits up by the sum let me use S and T let me use (())(28:31) simple spaces of S, so we say u plus v such that u will be the range of X and v is the range of Y, that is how any element to the range of Z looks.

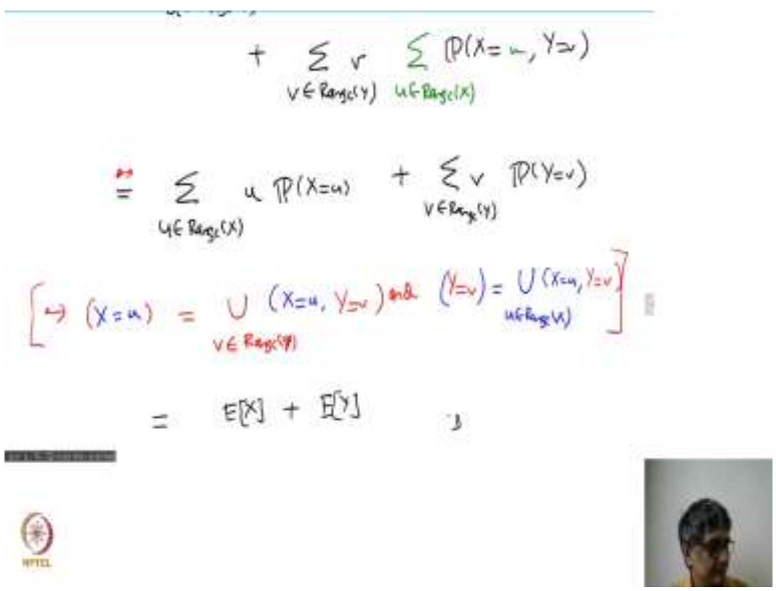
So, before E of X plus Y, so is the same as think of it as E of Z, X is the same as the sum over all Z in the range of Z, Z times the chance that Z equal to z little z. And that is what is going to be, that is going to be equal to see replace Z by X u plus v, so same as the sum over all view in the range of X and all v in the range of Y, the chance of for you just put instead of Z you just put u plus v and then you just put X plus Y equal to u plus v. All I say I just say X equal to u and Y equal to v that is how I am let me just walk you through this again, let me just comeback, double screen up.

(Refer Slide Time: 30:20)

So, what did was I says I wanted to identify probability z is equal to z and the way one does is you try and understand the range of Z is just u plus V for which u range of X and v of Y, the E of capital E of Z this is going to be the sum over little z chance is Z equal to Z. But E Z comes as u plus v by u range of X and v range of Y, that means you can rewrite this sum as just u plus v and you can all split up as X is u and Y is v, that is the only ways Z will be u plus v, so once you do this splitting properly you are on your way. We come back again to the next page.

(Refer Slide Time: 31:09)





So, due to the same now as what now as the same as the sum over u and range of X, v the range of Y I can split the two sums up, so in the following way. So, I will write in this way I will do the since everything is finite I can rearrange anyway I want to rearrange so I will do this u (())(31:41)in the range of X, v in the range of Y I have u times the chance that u plus v times the chances that X equal to u and Y equal to v, so I can do all this because I will rearrangement when it is okay as series converges absolutely.

Otherwise, I cannot do this things, so as the same as the sum u in range of X this now you can split up, so now you do as is the same as, so let me write this down as I am keeping I read as probability X equal to u, so this I will move the v sum inside, so v will be sum plus I will think of the other guy has a sum over vth range of X, v range of Y times v then I will think of it in green, this range of Y (())(33:36) X the sum over in u range of X, then this is just the chance that X is equal to u and Y equal to v.

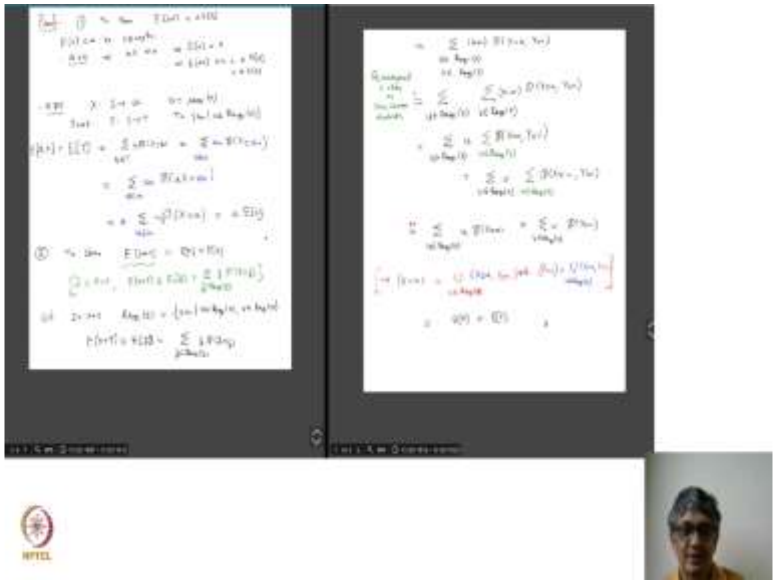
Now, I am in business now, now I know that if I sum over all values of Y interval I get the whole sample space, we all disjoint events, so the first thing becomes the sum over u range of X (())(34:07) equal to the sum over u range of X this is u and this whole thing is just the probability that X is equal to u and here are sum over the range of Y, v times this whole thing is going to be a probability that Y is equal to v.

And why is that why this is true? Why is this this equality true? Let me write it down why it is true? This is true because you know this fact that we know that even X equal to u is the countable disjoint union of v in the range of Y, let me write this in blue (())(34:55) X is equal to u is the same as X is equal to u and Y equal to v. that is just the same thing.

That is why this sum is true. And also and the reverse is true there is if you take Y equal to v that is the same as the event where you takes union over all possible (())(35:29) u in the range of X we will move this little bit, u in the range of X of X equal to u and Y equal to v that is just a fact just because that is why you can just remove the sum up.

This equality is because this. And these are disjoint events and the countable union probably sums up. So, therefore but then now you are done, because this guy is just the same, go back to black, this guy the same as E of X that is the first sum as E of Y is. let me go back to my split screen I have one comment to make before I finish the proof.

(Refer Slide Time: 36:46)



This one I will make it small, you can see everything in this page, so here is one small thing I did not say, so at this step what we should do is we assume that everybody is well define, so we assume that to show this we first assume that it is well define, so let me erase that I will come back to this a little later. So, in this step you assume E of X plus Y is finite, E of X is finite and E of Y is finite.

So, you assume that then you do this whole computation. So, there is a small chance that which I have not shown here that if E have X is finite and your Y is finite, it could be that E have X plus Y is not finite, but that is something that one can sort of work out and show that first that X plus Y is random variable, if you also would have finite expectation, and that is because it can work backwards from the last equality and we are in the sum showed series (())(37:48) absolutely. So, (())(37:52) shown the fact that E of X is finite. Let me here stop today I will come back and finish the proof for the second part third part on the next lecture.