Introduction to Probability with Examples Using R
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Lecture No. 21
Expectation of Random Variables

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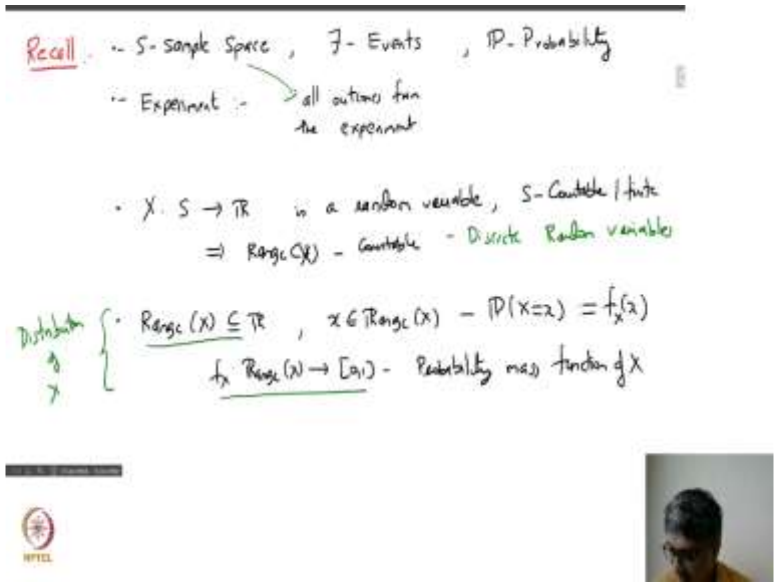

I will started today's class, so until now we will doing a so various random variables and try to understand there distributions coming to expect. So, let me just recall little bit, so we recall let me recall, so what have we been doing? We have been taking we have start off with a sample space S, so S was the sample space, we had and this we thought of as all possible outcome of an experiment.

Then we had what is called an events instead all events as or now we took events as all subsets of S, then we had a P which a probability on sample space and now what we did was we said, fine, now so each these was connected to an experiment and sample space was thought of as all possible outcomes to the experiment all outcome that comes from the experiment.

And you can think of the experiment as selecting one outcome from the sample space and events are point we are interested in and P is the chance that a particular event but after (())(02:01) that is one thing, then we say okay I had to keep on changing sample spaces again again, so I define this concept of random variable this function from S the real line, it is a random variable, so as if now I have not define it for all possible set S, so I will think of it S as a well S is discrete an S is countable or finite.

Then this will actually imply that the range of S also is countable range of S is also countable and all these added variables are discrete variables. Now, the next thing we did was it is looking fine, now each random variable is characterized by two things, so one was the range of X, so some subset of the real line, the other thing was how it what for any X the range X you wanted to know what the chance that X will be equal x and this we can define as what is call the probability mass functions, this we called as this function f from f of x from the range of X to 0, 1, we call it as the probability mass functions.

And the whole idea was that this essentially if I know the range if I know the chance that how each values on the range are distributed by X this would give you what is called the distribution of X and this should be enough to understand the reason. So, now I want to go back and see, this is one way of understanding random variables, you know it is complete distribution so you know exactly what is going on, what value it takes (())(04:39) it is now what to do build on this little bit whatever we do is I want to now go back to an earlier question that we ask long (())(04:50)

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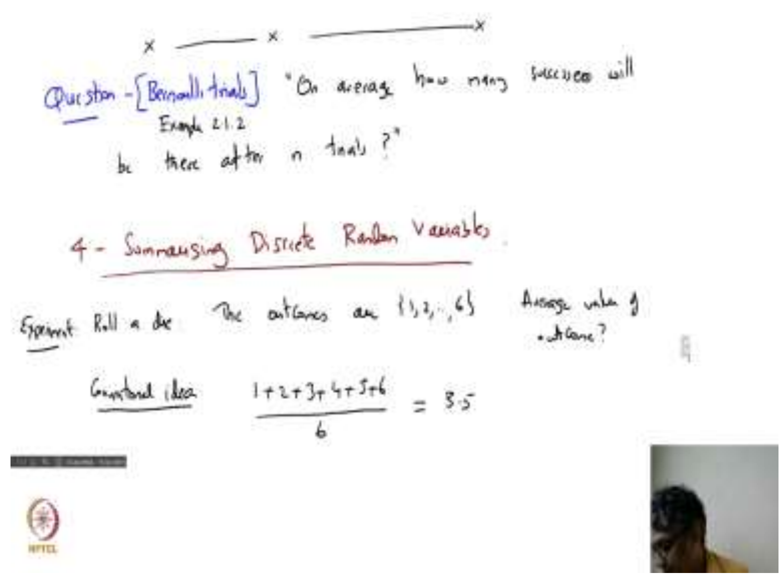

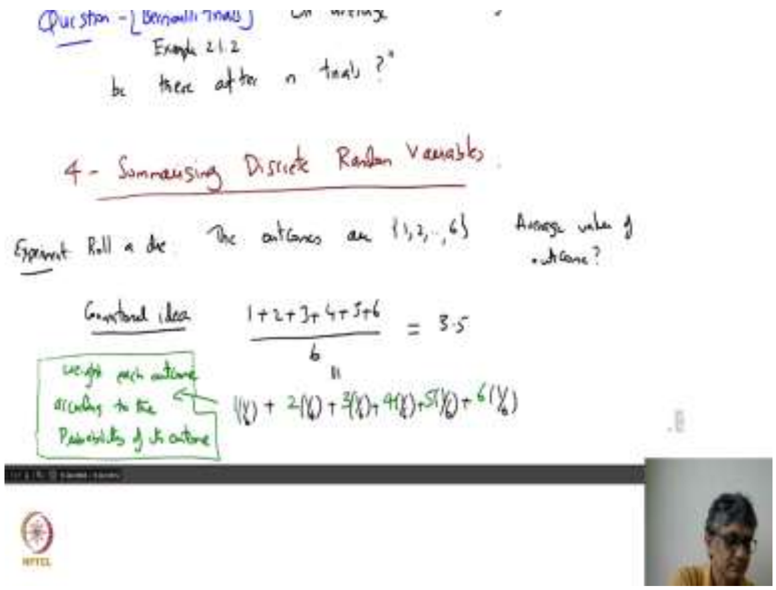

So, question I asked now came back to the following I said I know a few we recall this question way back into a Bernoulli trials I ask this question I said I think this is an example 2.1.2 say example 2.1.2, I will ask this first question I said on the average how many successes will be there after n trials? And this is way I ask you question to ask you know that you know the chance the (())(05:57) given by the binomial that is easy to understand.

But how do you understand this que stion? How many success will be there on average? So, this is what we will try and do now, we will try and do, this is the focus of this chapter which is called so the key what I try and understand is the key summaries I just call it as chapter 4 I will be summarizing discrete random variables.

So, this is the topic of this session, so let us go back to a very old idea that we have let us that is so suppose I give you a bunch of numbers, so h ow do you find an average (())(06:55), let us say we roll die, so we have a so let us say we roll a die, let us go back an old idea and experiment we know the die the outcomes are 1, 2, up to 6, so then you want to know what is the average outcome, let us see what is the average outcome.

So, the only one would be you just what you do is you just take the so if we have numbers 1 to 6 we all know our conventional idea is the following, it is 1 plus 2 plus 3 plus 4 plus 5 plus 6 by 6, that is the average number and this we would computed to be 3.5, that is what we would get. So, then you can just re-write this calculation little bit, let us see how one would do that, let me do this.

So, what it one would do is, one would split this up, as 1 as the value of the random variable takes on the role of the die takes 1, 2, 3, 4, 5 and 6 and then I would think of it as 1 into 1 by 6 plus 2 into 1 by 6 plus 3 into 1 by 6 plus 4 into 1 by 6 plus 5 into 1 by 6 plus 6 into 1 plus 6, so the same as the talk, there is no different, as re-regions of whole. So, now what did this perspective give you?

This perspective says that you weight each outcome of the roll of the die with the according to the probability of its occurrence outcome according to the probability of its outcome and that is what this is line is say. So, now the idea is that if you take a bunch of numbers into their average you are giving the chance that every number appears equal in your list with the same as the role of a die experiment.

But what happens if you have a list of numbers where every number does not appear equal? Like you may have random variables like binomial distinct different values for different something we will try and understand that this is a motivation so what we could do is we could take a list of numbers and if you have probability distribution of that the average value would be the weighted sum of those numbers and the ways being to the probability of the block. So, now that comes to a first definition that was first definition let me write down the definition.

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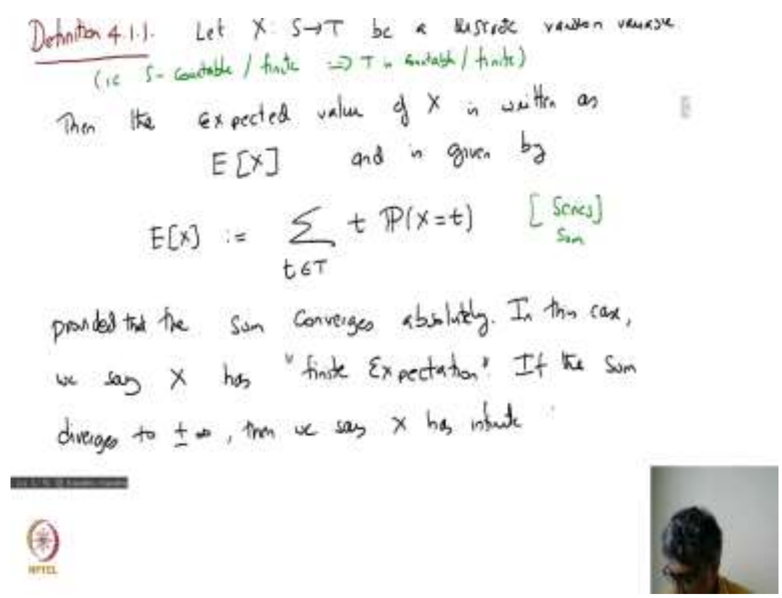

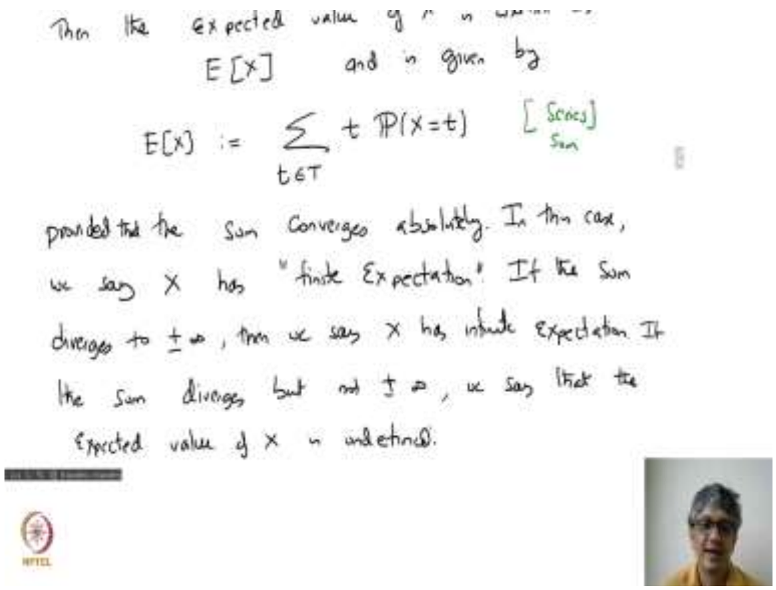

So, here is definition 4.1.1. so, here let X be from S to T be a discrete random variable, so automatically I know that X S is discrete that is for confirmation that is S is countable or finite, so it is automatically imply that T is countable or finite. And what you want is you want to know the expected value so then the expected value of X is it written as E of X, it is written as E of X, so I use square brackets E of X and is given by so the formula the following, so what you do is, you sum over all possible values X takes values in, so range of X is in.

So, what you do is you sum over all possible values in T and then you multiply the value t and the way which give the sum is particular number is the chance that X is equal to t. Of course, this same is if t is finite is well defined, otherwise it is a series, so you have to carry at by saying that this is equal to this provided the sum converges absolute. So, provided that the sum converges absolutely.

So, in this way in this case in this particular case when the sum converges absolutely, we say X has finite expectation, X has I will write it has finite expectation. Now, whereas so line is clear this sum converges we say X has finite, am I still audible can we get start? Give me a chat link. Thank you. So, now what happens if the sum diverges, so the sum at this point c sum so the sum diverges then we will say that the random variable has, not has does not actually so they define it is that if the sum diverges it does not emerge but it does not go to infinity but not to infinity.

So, the series cannot diverges infinity, (())(15:22) we will say so maybe I will go first like this sum diverges to infinity so we will do that first, so sum diverges to infinity we say that the random variable has infinite expectation, so sum there is a plus or minus infinity then we say we will say X has infinite expectation in both cases.

Because the sum diverges neither to plus infinity to minus infinity we say that the expected value is not defined, so if the sum diverges but not to plus or minus infinity, so it does not go anywhere just oscillates we say that the expected value is not defined, expected value of X is not defined. So, that is the idea of random variable X, then you define by the sum the sum converges (())(16:52) that the expect value you see the finite expectation, if we diverges plus or minus infinity you say that the random variable infinity expectation. And the sum diverges but not to plus minus infinity this are the expected value of X is undefined. So, let us see a example the simple one just to illustrate the idea.

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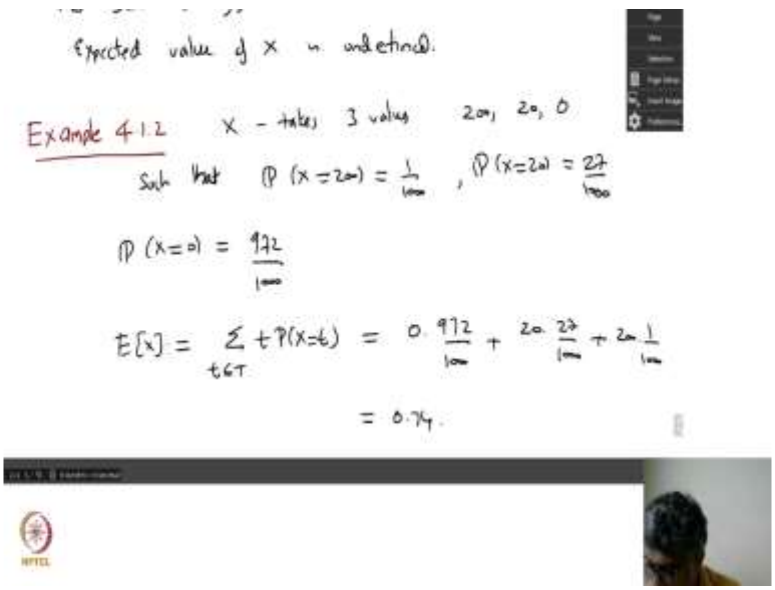

So, here is an example, example 4.1.2. So, let us say a random variable takes three value, let us say X is the random variable X, X3 value (())(17:51) X takes three values let us say 200, 20 and 2 or 0 let us say, 0, such that the probability of X equal 200 is equal to let us say 1 over 1000, X equal to 20 say 27 over 1000 and X can take the rest high probability X is equal to 0 with 972 by 1000.

Then this would mean that the expected value of X by definition is the chance that is what the chance that t is it T, t times probability X equals to t, so what you do is you just multiply it by so you just have 0 in the first one the probability which takes is 972 by 1000 plus 20 times 27 by

1000 plus 200 into 1 by 1000, so we add this up you will get some points on, so an average X will take 0.74 notice that this number is not attain by X, let me just explain this concept little bit in view let us see if I can get this thing to screen to come up again.

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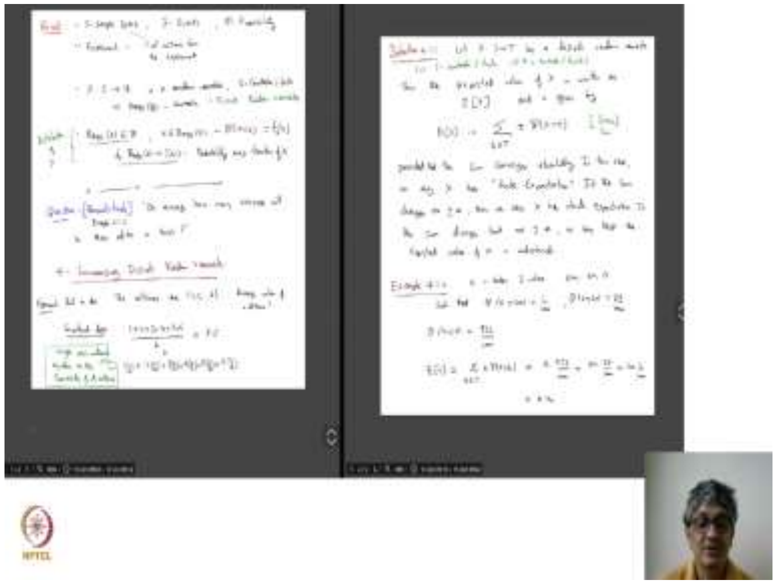

Of course, I want to put the first page here, (())(20:00) second page, so this is what we been doing. So, we start out with the idea that what is the sample space, sample space S is this (())(20:19) experiment and we had random variables and we understood them by distributions, but then we want understand on a average what value random variable will take, the (())(20:29) that much saying that you become die experiment the number of 1 to 6 and then if you want to find the average of 1 to 6 the conventional idea would be you sum them up and divide by 6 and then you 3.5 but if you put you observe that little bit that just says that (())(20:46) 1 over 6 to every number in the list.

And that gives you a number 3.5 and this motivated us, the definition of average for any random variable, (())(20:59) finding X goes from S to T, then S is countable, then T automatic countable so you have discrete random variables that the mean of X for the expected value of X would be written as sum from T, in T value T and the probability that X takes the values and that weighted sum is called the value of X.

And the sum is a countable sum, so it can converge or not converge or diverge infinity. So, in these three cases you define it as if it draws absolutely you say it has final expectation and the sum is nicely defined, if the sum diverge in plus minus infinity, you see the random variable as infinite expectation, if the sum does not diverge to any number we say they are diverges to plus or minus infinity then we say that it is the value of X. Now, let me go and find let us have some properties of this one of E of X. So, the first thing in the first theorem is that if random variable takes one value all the time let us just proof that case up. Let me do that summation.

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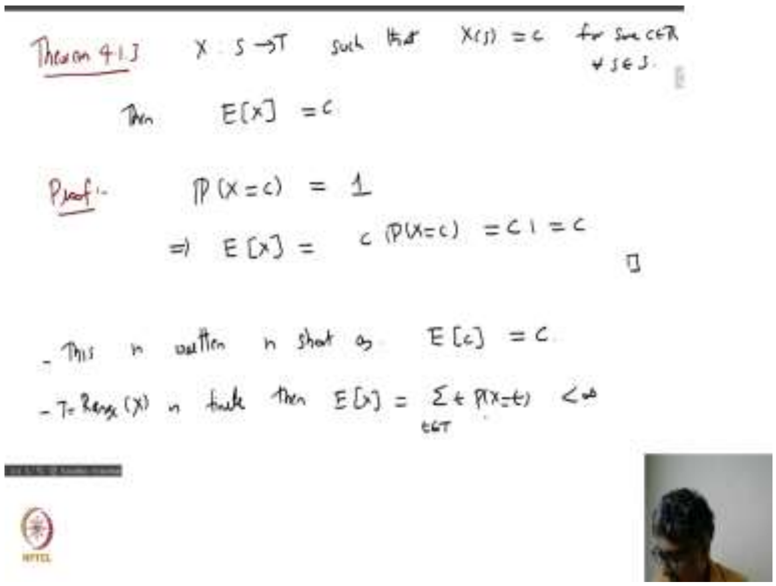

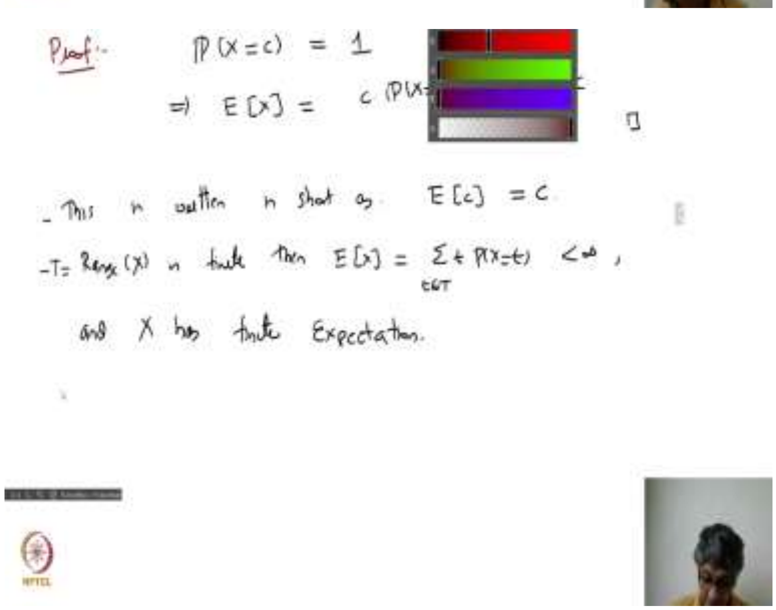

So, here is the theorem, I called it 4.1.3, the idea is the following. So, let X be a random variable S to T such that X of s is always equal to constant c for some c for all S, then expect value of X is just going to be equal to c, how is that how the proof is very simple, so when we had a proofed up so proof it involving that you just take you know that the chance that X equal to c is equal to 1.

The X want to take one value c, then this would imply that E of X is just c times the probability X equal to c which is just the very simple completion see that 1 which is equal to c. So, a constant random variable will have its expression as c, so this is written n short as E of c is just c. So, expectation of random variable is constant value, I put C n for the random variable who added as E of c. So, that is one thing is this one degenerate cases (())(24:28).

The other observation you can also make is the following suppose the range of X is finite so range of X is finite, X is finite say T is equal to that X is finite, then E of X is always finite (())(24:52) E of X is going to be summation t t and T P probability X equal to t is always going to be finite, because the finite sum is always going to be finite so we will say that X has finite expectation and consequently and X has finite expectations. So, when the range of X is infinite as countable as countable to infinite then there is a possibility that it could have finite or infinite prediction or it not had to be consequences. So, let us see two examples, whether this is not valid.

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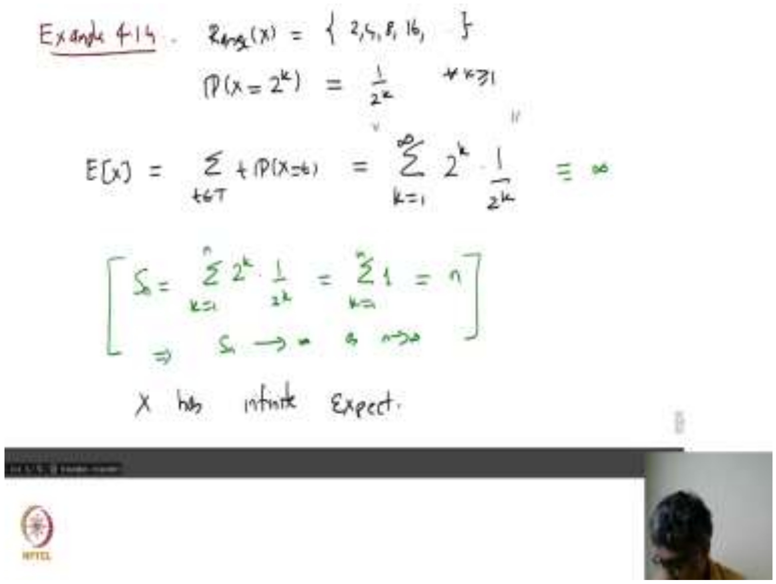

So, here is the example, so idea is this, suppose X takes the value let us say the non-negative even so let us say range of X is all the even intervals let us say 2, 4, 8 all multiples of 2 and let us say it takes the chance that X is the probability mass function X equal to 2 power k, let it be equal to 1 over to 2 power k, for all k.

So, it is clear that this is a probability, so then what is the expectation X, the expectation of X in this case would just be the sum over t, t times probability X equal to t that I can rewrite as the sum over k equal to 1 to infinity of what, so I have 2 to the k is the value that the thing takes and then the probability here it takes is 1 over 2k.

And so this of course I know we will do properly let us say how do you understand this sum, so they what we do it is (())(27:33) let us say S sub n equal to the sum from k equal to 1 to n, so let us go back little bit so 2 to the k and 1 over 2 to the k that is the sum from k equal to 1 to n of the sum number 1, there are n times you are adding 1 so to the n, so this will imply that Sn would diverge to plus infinity as n.

That means this number is going to be infinity, so that means the random variable has infinite, so this means that X has infinite expectations. Let us, do other example where this where we can see that there are some examples, where we do the following which say that let us say I give the rule just to get the colour line, so another example.

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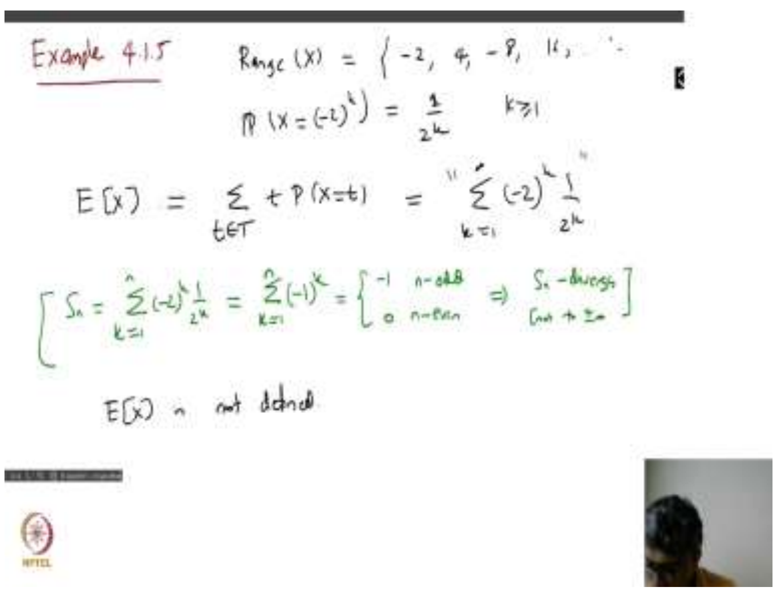

So, example say 4.1.5, black, let us see this where here suppose X variable takes let us say range of X is let us take the negative power so let us say minus 2, then 4, minus 16, plus 16, then is it minus 2 to the power k, whole power, that means I do same thing same distribution I say X is this number which is minus 2 to the power k it takes the value with 1 over 2, it is obviously a probability for k equal to 1 because the sum upto 1 geometric series.

Now, if you want to do expect value, so it is value of X we would define it again as the sum from t and T chance at P and then X equal to t. So, here again you would write it formally as the sum from k equal to 1 to infinity minus 2 to the power k 1 over 2 to the power k, so again what is the sum mean, the sum means what? The sum means would again do a following say Sn is equal to the sum from k equal to 1 to minus 2 to the power k and 1 over 2 to the power k.

And this we know exactly is going to be the same as the sum from k equal to 1 to n, the 2 to the k catch or drop and it left with minus 1 to k and this we know if you take the starting minus 1 and if adding 1 and minus 1 you get 1 minus 1 for n is odd and 0 when n is even. So, this will imply Sn diverges and not to plus or minus infinity because it keeps oscillating.

So, that means that it just conclude that E of X is not defined. So, the expectation of X is undefined, it is sort of nice interesting fact let us see if I can also know basically I will write this theorem down and then may be thing a little bit how to show it. So, the lines interesting a useful mathematical tool will keep on using it sometime.

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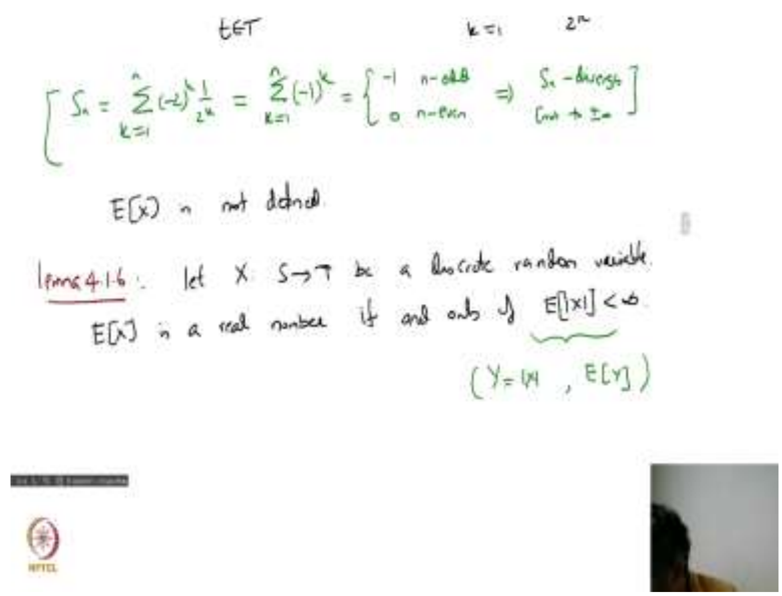

Here is a lemma, the lemma is the following. So, where is E of X here now? So, E of X let X be random variable be a discrete random variable, then E of X is a real number if and only if E of mod X is finite. So, is this for what any statement is that I whatever statement so, X is random variable, so this person here what I mean by this statement, I take a random variable Y which is equal to mod X, the actual value X and this just I am just considering E Y.

So, if E Y is finite then I know that E X has to be 1 and vice versa. The nice interesting fact probably that is a random variable has fine expectation if and only if E of mod X is finite, so one way should be obvious I think if E of mod X is finite and E of X is finite, but that is let us just try and work out the proof properly. So, maybe I will just one small thing before I discuss we will go to my split view again. Very nice.

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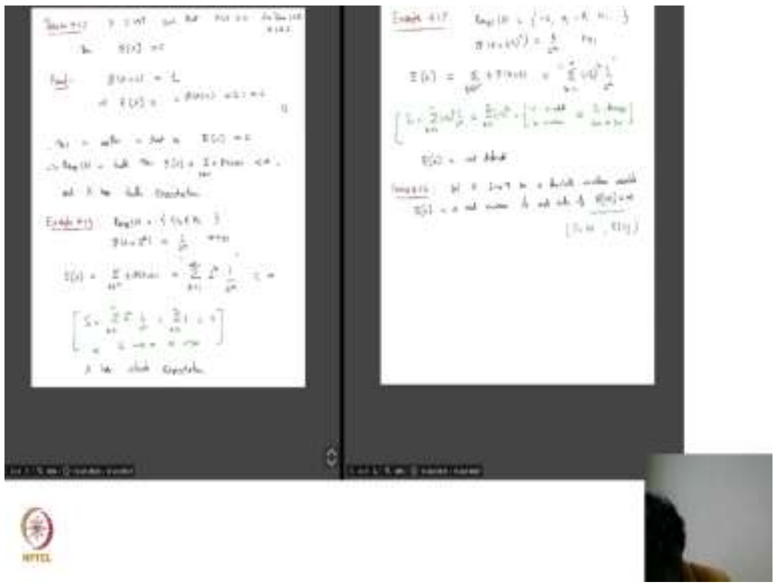

So, to the previous page still I want from here. So, here again I wanted to summarize what I teach so far, so X is constant then E of X is constant, so you write that as E of c is equal to c, the random variable is finite and obviously as my expression, if it is countable then you have two cases the first case in example 4.1.4 we found out that the mean diverge infinity, the second case we said that we saw that take the values in let me write this, If it took values minus 2 to power k and it keep oscillating will make it positive the mean would not well defined.

And say E of X is defined, let us see diverges to plus infinity, similarly now when can you see a random variable is finite mean one condition is that we if the absolute value has finite as (())(34:41) Y equal to mod X and take E of Y is finite then we can say E of X also is finite. So, now this I will this conclude this today in the first class by the following that when you what do you mean by E of X, say physically? Suppose I like if, so one thing think about all this fine or this formula are fine, but one thing we will try into the lemma next in the next class, we will proof the lemma next class.

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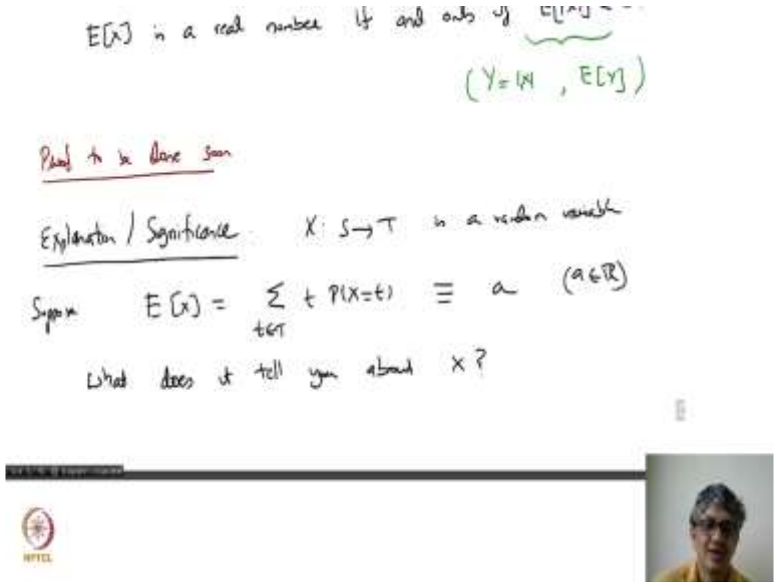

So, proof could be done soon, but I would not understand is that is let us see the explanation for let us say significance, so we spend a minute or so on, suppose I have let us say X random variable x should be random variable and suppose let us say a (()(36:02) and say E of X is 4 let us say E of X could be the sum from suppose t and T probability X equal to t is some number is a is some number is a some way or the other.

Then what does it mean? What does it tell you about the random variable? So, that is something to think about. So, what does it carry? So, your E of X is just a number a, what does it tell you about X? Something about you think about it, because it is a formula I know and also like you but you can go back to the old definition we had, 1 through 6 we add the numbers up we get 3.5. So, what is the average value of a bunch of numbers? Of for example for your class, for example if I tell you that the mean number means score of the last quiz was let us say 10 out of 20, what does that mean? You it gives you some idea, so that it should relate to what a random variable mean is or expected value.