

Introduction to Probability with examples using R
Professor. Siva Athreya
Theoretical Statistics and Mathematics Division
Indian Statistical Institute, Bangalore
Lecture No. 20
Functions and Independence

(Refer Slide Time: 0:15)

The slide contains two columns of handwritten mathematical derivations. The left column shows the derivation of the PMF of $Z = X + Y$ where X and Y are independent Poisson variables with parameters λ_1 and λ_2 . It starts with the convolution formula $P(Z=n) = \sum_{k=0}^n P(X=k)P(Y=n-k)$ and proceeds to show that $P(Z=n) = \frac{e^{-\lambda_1} \lambda_1^k}{k!} \frac{e^{-\lambda_2} \lambda_2^{n-k}}{(n-k)!}$, which simplifies to $P(Z=n) = \frac{e^{-(\lambda_1 + \lambda_2)} (\lambda_1 + \lambda_2)^n}{n!}$. The right column shows the derivation of the conditional distribution $P(X=k | Z=n) = \frac{P(X=k)P(Y=n-k)}{P(Z=n)}$, which simplifies to $P(X=k | Z=n) = \binom{n}{k} \left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{n-k}$. A small video inset of Professor Siva Athreya is visible in the bottom right corner of the slide.

So, now the I just want to quickly recap what we did, we were doing so far. So, then the idea was that in we start with the idea that how to understand sums of random variables. So, we did the example of a to Poisson guys with independents. So, left hand side we immediately see that this let me take independent the sum is given by the combination the distribution of sum is going to convolution and once we have that then we are able to work out that an interesting fact that if Z Poisson adds up, so if X was Poisson lambda 1 one the lambda 2 with a independent then Z was Poisson lambda.

Then you can generalize this, this prove is by induction if you have k random variables there are all independent plus sums and then we also Poisson lambda i in then the Z gives you Poisson of the sum then we come here and then we notice that X Poisson lambda 1 Y is Poisson lambda 2. And I ask you with distribution of X given Z equal to n. We did this calculation using the definitional conditional property and we found out known behold if the chance that x is equal to k given Z equal to n was n choose k lambda 1 by lambda plus 2 equal power k times 1 minus c in fraction then we got n minus t.

(Refer Slide Time: 1:46)

The image shows two columns of handwritten mathematical derivations on a whiteboard. The left column derives the binomial distribution from the sum of independent Poisson variables. It starts with $X = X_1 + \dots + X_n$ and $Z = X + Y$, where X_i and Y are independent Poisson variables with means λ_i and λ respectively. The derivation shows that $P(X=k | Z=n) = \frac{P(X=k, Z=n)}{P(Z=n)}$, which simplifies to $\frac{e^{-\lambda} \sum_{k_1+\dots+k_n=k} \prod_{i=1}^n \frac{\lambda_i^{k_i}}{k_i!}}{e^{-\lambda} \sum_{k=0}^n \frac{\lambda^k}{k!}} = \frac{e^{-\lambda} \frac{\lambda^k}{k!} \binom{n}{k}}{e^{-\lambda} \frac{\lambda^n}{n!}} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$. The right column states a theorem: 'Theorem: If X_1, X_2, \dots, X_n are independent Poisson variables with means $\lambda_1, \lambda_2, \dots, \lambda_n$, then $Z = X_1 + X_2 + \dots + X_n$ is a Poisson variable with mean $\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$. For any event B in the sample space, $P(f(X_1, \dots, X_n) \in B) = P(Z \in B) = e^{-\lambda} \sum_{k \in B} \frac{\lambda^k}{k!}$.' Below the whiteboard is a small video feed of a man with glasses and a yellow shirt.

And this we figure out was the same as saying that X given Z equal to n was binomial n of λ/n . So, an interesting fact if you have two Poisson your conditional sum then the individual Poisson becomes a binomial, so it is an interesting fact. And this whole this understanding distribution of function will be add in the following theorem is which have been using computationally on the example is that if your function and random variables then all is in the factors that then the function of x_1 to x_n lands in event P is the same as x_1, x_2, x_n landing in the event f inversely. And once you capture this idea and everything as follows by is equal to. Let us move on to the next topic.

Student: Sir, can I just ask you a small question?

(Refer Slide Time: 2:53)

3.3.2 - Functions and Independence

If X and Y are independent random variables

$$X: S \rightarrow T_1, \quad g: T_1 \rightarrow \mathbb{R}$$

$$Y: S \rightarrow T_2, \quad f: T_2 \rightarrow \mathbb{R}$$

$$Z = g(X), \quad W = f(Y)$$

(intuitively) $\Rightarrow Z$ and W are also independent



$$X|Z=n \sim \text{Binomial}(n, \frac{\lambda_1}{\lambda_1 + \lambda_2})$$

Theorem 3.3.5: Let X_1, \dots, X_n be random variables on a single sample space S . Let f be a function of n variables, $Z = f(X_1, \dots, X_n)$ is well defined on S .

$$P(f(X_1, \dots, X_n) \in B) = P((X_1, \dots, X_n) \in f^{-1}(B))$$

for any event B in the range of f

Proof (Sketch):

$$\left\{ s \in S \mid f(X_1(s), \dots, X_n(s)) \in B \right\} \quad \text{let } Z \text{ be } f \text{ set}$$



Professor: I want move to the split view split view is good. So, I will not understand this idea of functions and independence. So, it is a 3.3.2 is functions and independence. Sure.

Student: Can you hear me?

Professor: Can you ask it on chat is it possible? Let me carry on I will come back and answer to the question you asked. So now, so there was a question on the previous slide let us go back again let me repeat the answer so the argue was that, so we had shown in this computation that if X Poisson lambda 1 Y was Poisson lambda 2 and Z was X plus Y then distribution of X given Z equal to n was given by binomial comma lambda.

So, what I meant to say in terms of intuition of the following that this is the competition we have done. So, X given Z equal to. So, and the understood Poisson in this class so far as a limiting distribution of binomials. But the idea is that there is a interpretation of Poisson that you can use in the acquiring model where this interpretation on X given Z equal to n being binomial is very useful that is all I meant to say.

So, it is little bit outside the scope of this class but we will come and see there. I will try and see if I can go to any at some point family. Now let me go on to function and independence. So, now here is the idea is the following if X and Y are independent random variables. And then I have function f let say X takes value from S to $T1$, Y takes values from S to $T2$ then I have function g from $T1$ to R I have function f from $T2$ to R .

I look at Z equal to f of x sorry g of x I look at W equal to f of y . So, there is no reason to believe so if X and Y are independent there is no reason to believe that Z and W will not be independent. So, this should actually imply that let me write this limit better so it is clear. Let me go back to black. So, g of x and W is equal to f of y there is no reason to believe that Z and Y are not independent. So, this will actually imply Z and W are also independent.

Because W depends on Y alone Z depends on X alone if X and W are independent X and Y independent then Z and W should be independent. This is something that is all classical into intuiting kind of make sense, so this into intuiting make sense.

(Refer Slide Time: 7:31)

Let $n > 1$ be a positive integer. For $j \in \{1, \dots, n\}$ define a positive integer $m_j \in \mathbb{N}$. Suppose $\{X_{ij}\}_{i \in \{1, \dots, n\}, j \in \{1, \dots, m_j\}}$ be an array of mutually independent random variables. Let f_j be functions such that $Y_j = f_j(X_{1j}, \dots, X_{m_j j})$ are well defined. Then the resulting random variables Y_1, Y_2, \dots, Y_n are mutually independent.



So, let me write a theorem down let us go and classify this properly. Theorem let n bigger than or equal to 1 be a positive integer. So, I take bigger than 1. So, let me add at least two random variables in my collection, so I write the general theorem down let us say I take j in the event 1 up to n I will set one of them.

Let define for j in this set 1 of 10 and define a positive integer m sub j . And suppose I have X_{ij} . So, X_{ij} is what? j is in 1 to n so j goes from 1 j is in 1 up and i is from up to n subject. So, be an array of mutually independent and a variables for j in the 1 up to n . So, this already is I already said this no need to save this. Then I let f sub j be functions such that f sub Y_j is f sub j of X_{1j} up to X_{nj} are well defined.

So, that means f sub j is well defined on this collection of f plan and variables are X_{nj} to X_{mj} so each of j acts on different spaces X of m_j is perhaps to real line. Then so they are all independence so this crucial they are all independent variables independent collection then if I look at y sub j are also independent. Then the resulting random variable Y_1, Y_2, Y_n are mutually independent. So, general fact about random variables that ones can sub use in sense that.

So, general theorem one can write down so in terms of function and a variables. So, if you have a random variable in array, then and the array is define as such a way that your separate functions and in acting on independent part of the array sorry and disjoint pass the array. Then the resulting sequence is also a mutually independent. So, general theorem now this side up to down tables. We will see recommend of two down tables.

(Refer Slide Time: 11:39)

Proof:- "Random quantities produced from independent inputs will be independent".

let B_1, \dots, B_n be sets in the ranges of Y_1, \dots, Y_n

$$\begin{aligned} & \mathbb{P}(Y_1 \in B_1, \dots, Y_n \in B_n) \\ &= \mathbb{P}(f_1(X_{11}, \dots, X_{m1}) \in B_1, \dots, f_n(X_{1n}, \dots, X_{mn}) \in B_n) \\ &= \mathbb{P}((X_{11}, \dots, X_{m1}) \in f_1^{-1}(B_1), \dots, (X_{1n}, \dots, X_{mn}) \in f_n^{-1}(B_n)) \end{aligned}$$

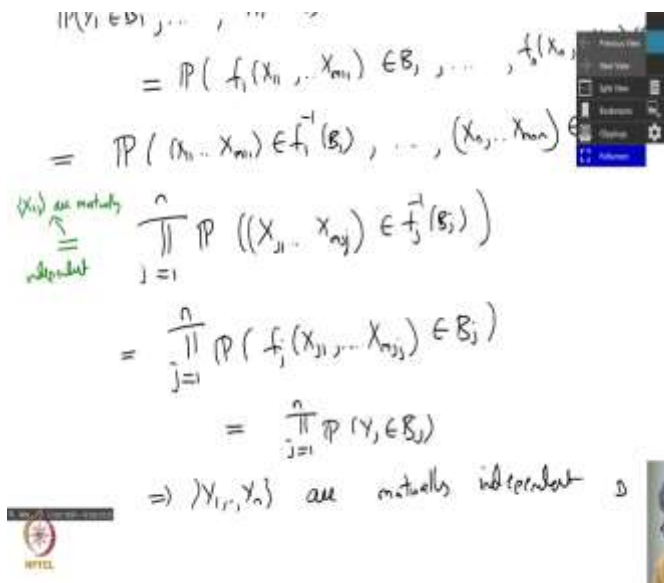


So, proof of this so what is prove in this? So, the prove is a so when add intuition profit theorem what the theorem saying with you the theorem is saying intuiting that random quantities. So, the theorem if I have to state in words random quantities produce from random in from independent inputs will be independent. That is what this theorem is saying. So, then up to for this so let B_1, \dots, B_n be the sets in the ranges of Y_1 through Y_n .

So, then what I have to show I have to that the chance that Y_1 in B_1 all the way up to Y_n and B_n this splits up as a problem. So, what is first line let see so here is interesting catch I will leave it as an open question for you guys to understand. So, the first thing is we write down this that is easy, the first thing is that watch Y_1 , Y_1 is f of X_{11} up to X_{m1} this is in B_1 and all the way up to f_n of X_{1n} all the way up to X_{mn} is in B_n , that is what this is of this is the same as the chance that if you look at each of these events look at X_{11} up to X_{m1} is in f_1 inwards of B_1 all the way up to X_n X_{mn} and in f_n inwards of B_n that this theorem.

Now you know that the X_i . So, the X_i is an independent X_{ij} are independent are mutually independent. So, this becomes a product of j equal to 1 to n the probability of each of these of guys. So, you take X_{j1} , X_{mj1} so m_{jj} . Let me say it is a Y let me just do the reverse in this. Because notation that will be say i equal to 1 to n . Let me just go at this matter m is equal to j , so j equal to 1 to m X_{jn} at n_j m_{jj} is in the event f_j inverse of $B_{sub j}$ this product.

(Refer Slide Time: 15:23)

$$\begin{aligned}
 & P(Y_1 \in B_1, \dots, Y_n \in B_n) \\
 &= P(f_1(X_{11}, \dots, X_{1n_1}) \in B_1, \dots, f_n(X_{n1}, \dots, X_{n, n_n}) \in B_n) \\
 &= P((X_{11}, \dots, X_{1n_1}) \in f_1^{-1}(B_1), \dots, (X_{n1}, \dots, X_{n, n_n}) \in f_n^{-1}(B_n)) \\
 &\stackrel{\substack{\text{X}_{ij} \text{ are mutually} \\ \text{independent}}}{=} \prod_{j=1}^n P((X_{j1}, \dots, X_{j, n_j}) \in f_j^{-1}(B_j)) \\
 &= \prod_{j=1}^n P(f_j(X_{j1}, \dots, X_{j, n_j}) \in B_j) \\
 &= \prod_{j=1}^n P(Y_j \in B_j) \\
 &\Rightarrow \{Y_1, \dots, Y_n\} \text{ are mutually independent. } \square
 \end{aligned}$$


Now you just go back lap where you get the same as product of j equal to 1 to n probability that f sub j of X sub j is in B sub j. And that is the same as the product of j equal to 1 to n probability is in that Y sub j is in B sub j. And this I have shown for any possible events B sub j's which proves that is we imply that Y1, Y2 to Yn are mutually independent. Let me quickly recap the prove will in goes double split view and show it both sides.

(Refer Slide Time: 16:35)

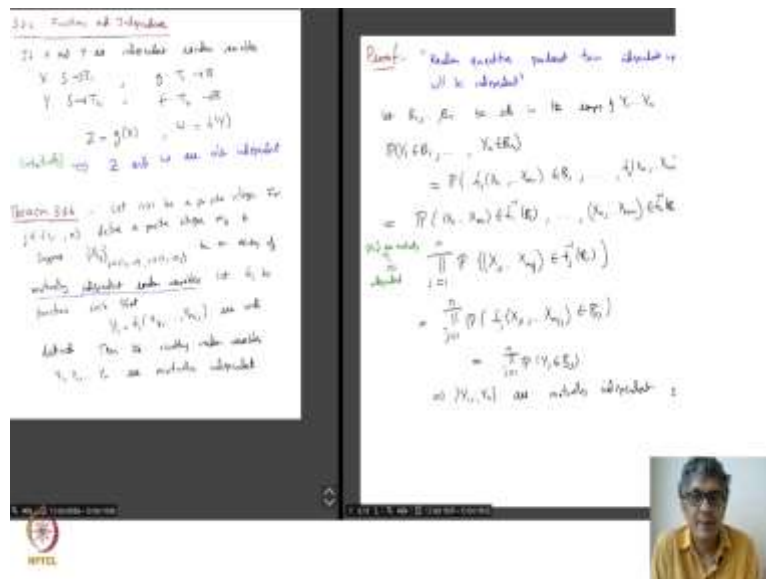
Def: Factorial Independence

Let X_1, \dots, X_n be independent random variables. Let $Y_1 = f_1(X_1, \dots, X_{n_1})$, $Y_2 = f_2(X_{n_1+1}, \dots, X_{n_1+n_2})$, \dots , $Y_k = f_k(X_{n_1+\dots+n_{k-1}+1}, \dots, X_{n_1+\dots+n_k})$. Then Y_1, \dots, Y_k are mutually independent.

Proof: "Each variable picked from independent will be independent"

Let B_1, B_2, \dots, B_k be sets in the range of Y_1, Y_2, \dots, Y_k .

$$\begin{aligned}
 & P(Y_1 \in B_1, \dots, Y_k \in B_k) \\
 &= P(f_1(X_1, \dots, X_{n_1}) \in B_1, \dots, f_k(X_{n_1+\dots+n_{k-1}+1}, \dots, X_{n_1+\dots+n_k}) \in B_k) \\
 &\stackrel{\text{independence}}{=} \prod_{j=1}^k P((X_{j1}, \dots, X_{jn_j}) \in f_j^{-1}(B_j)) \\
 &= \prod_{j=1}^k P(f_j(X_{j1}, \dots, X_{jn_j}) \in B_j) \\
 &= \prod_{j=1}^k P(Y_j \in B_j) \\
 &\Rightarrow \{Y_1, \dots, Y_k\} \text{ are mutually independent. } \square
 \end{aligned}$$



Clear? We will be doing this down little bit you can see the prove may be down you get let me go to previous page. So, this is what I trying to show. So, I said that functions are independent if this happens. If you have an array X_{ij} and you have functions acting on disjoint pass the array these random variables acting under the independent. So, this what is the prove is very simple but there are the small exercise to do in the prove that the green part has to justify little bit.

But I will leave this exercise for you think about and the prove follows that the Y_1 Y_n are also mutually independent. So, the idea is that random qualities produce some independent inputs will always be independent.