Introduction to Probability- With examples using R
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Lecture-14
Discrete Random Variables – Part 01

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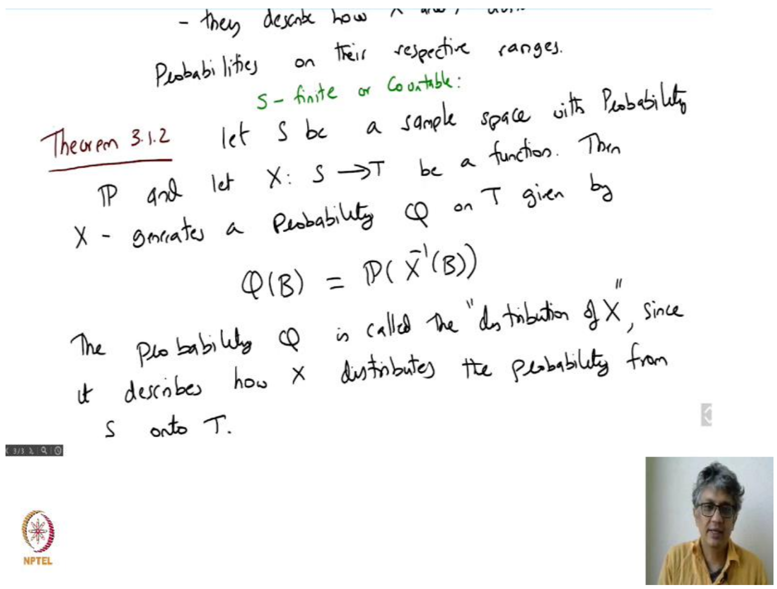

Let me start now so I was trying to do this theorem 3.12. So I will try and show the proof of this. So, the idea is that, you have a function X from a sample space to another place T. And then X generates a probability Q on T given by this formula Q of B is equal to probability of X inverse of B. And I have to show that this Q indeed is the probability. So, one assumption I always make in this whole sync, this whole section or chapter; is that I will always assume S to be either finite or countable. So in that sense, I will restrict my space.

So, number of outcomes in a sample space are discrete. So, now the idea is that I have to show that this is a probability. Let me try and do the proof of this. So what all do I have to show. I have to show first that in this, if I want to show a proof of this, I have to show first that, that the two actions are probable.

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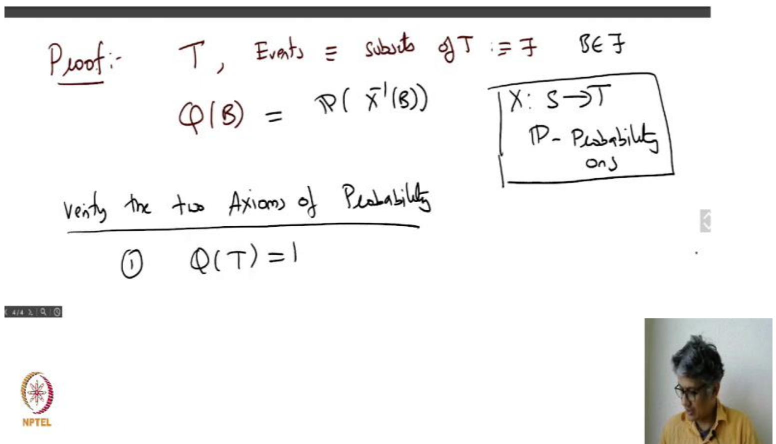

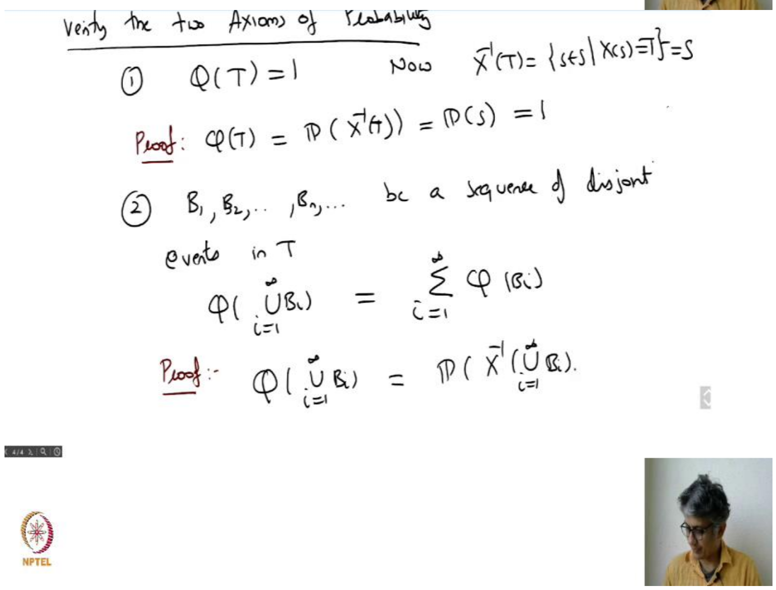

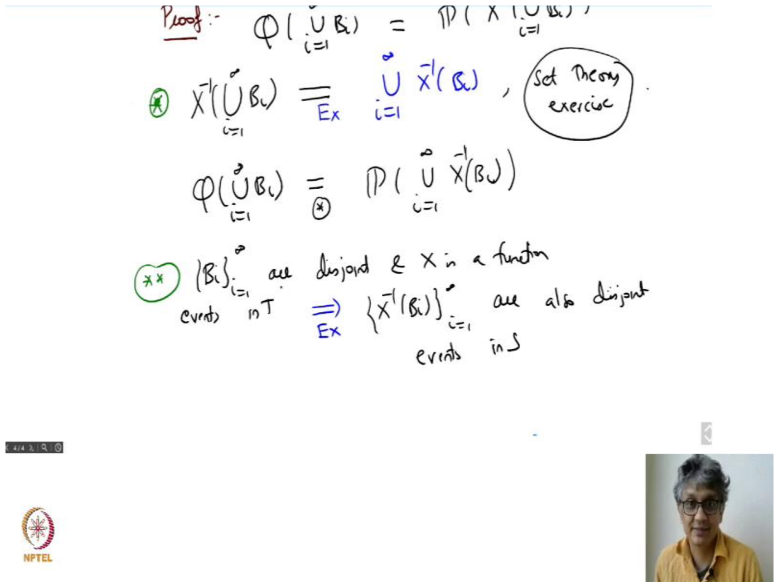

So what is the proof. The proof is, let me go here and change the colour little bit, to maroon. I will pull a bit more. Very good. So here the, the proof is the following. What is the proof? In the proof, I have to show what, I have to show two things. I have to show Q is the probability. So, Q is on so Q, T is a sample space and I have the events, are all power sets, are subsets of T. Events are all subsets of T. I take that and I have Q of B, any event B, which let us call this as F. I will do that with black.

So B and F. Q of B is equal to I define it as a X, probability of X inverse of this set. Very good. (())(02:23). So, now I am sure Q is a probability. So now, the first thing I have to do is I have to first show.So, what is X. So X is a function from S to T and P is a probability of S (())(2:36). This is I have given, this is given to us.

So, now what I have to show the first axiom of probability I had to verify is, I have to verify two things. So, verify the two axioms of probability. So, what are the two axioms. The first axiom is, the first axiom is I have to show Q of T is equal to 1. So, how do I verify that so what is Q of T. So, let us verify that. Lets verify that.I will write down in blue.

So that, that is, how do I verify that. So, Q of T is what Q of T is probability that of X inverse of T, but X is a function from S to T. So, everybody in T has to fall in S. So, same as probability of S and that is equal to 1, because X inverse of T is the set of all S and S such that X of S is equal to T. But X is a function of S to T, so this has to be equal to S (())(03:48).

Now, the second thing you have to verify is a little bit, a little bit tricky; but once you know the set theoretic ideas it should be easy. So, what you have to do. You know take B1, B2, Bn and so on and so forth be a sequence of disjoint events in T. And what you have to show. You have to show Q of the countable union of i equal to 1 to infinity of the Bi's is equal to the sum of i equal to 1 to infinity of the Q of the Bi's. This is you what you have to show.

Want to verify this axis. So, maybe here I will write it as, this is a different one, different colour. I do not know why this is again, again coming. So, this proof of this claim is 241 is here. Similarly, let me write down the proof of 2. So, what does proof of 2 give you. How do you show this. So, again let us look at the left-hand side.

So, Q of the union i equal to 1 to infinity of the Bi's. What is that going to be. That is the same as the chance probability of X inverse of the union i equal to 1 to infinity of the Bi's. Now, what is that now. This you will be a little careful. How do you do this now. So, you look at X union, X inverse union i equal to 1 to infinity of Bi. So, the little exercise which I want to show. It is a small exercise. One can show that this is the same as union i equal to 1 to infinity of X inverse of Bi. So, given that b and this is the definition. It is a set theoretic exercise.

So, once you have this you are in shape, because now you go down you say fine.This is the same as this is the first claim. So, the first claim, so this is, this is claim (())(06:30). This is a star. Let me call it as star. Then the next one is you have to go. So, what you do that means from star using star, using star I get Q of the union of i equal to 1 to infinity of Bi's is the same as the probability of union i equal to 1 to infinity X inverse of Bi. That is the next step.

Now, the next claim is the following that you do let me call this double star claim. This only you have to verify again. I will leave it exercise Bi i equal to 1 to infinity are disjoint and X is a function will actually imply. So, I will just do an exercise again, you can check about that X inverse of Bi are also disjoint, are also disjoint .

Disjoint events in S. So, X and we have disjoint events in T and X as a function will imply that X inverse Bi are also disjointed events in S. So, once you have this.

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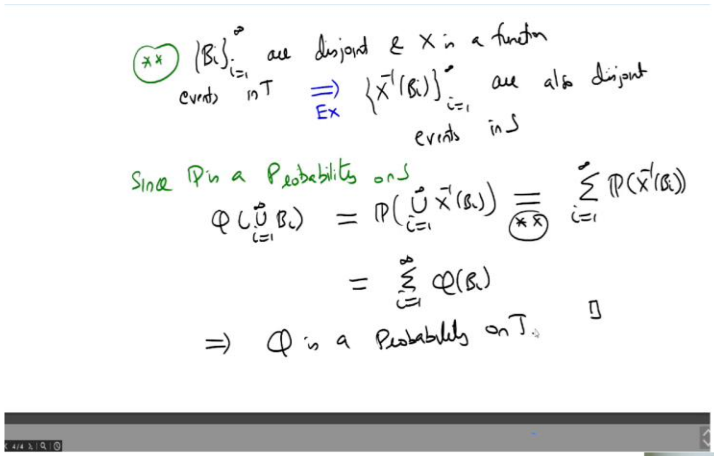

Since P is a. I will write it down in. Since P is a is a probability on S, probability on S. So, what do you get. You meet here on your way it implies Q of the union i equal to 1 to infinity Bi here I use double star. Now, that is not here the same as, is the same as the probability of the union i equal to 1 to infinity X inverse of Bi and already know this.

Now, since P is an i axiom of 2 and acts into a probability and double star will apply the same as probability i equal to infinity, probability of X inverse of Bi, because X inverse of Bi are the disjoint events in S. And that is the same as sum i equal to 1 to infinity Q of Bi. So, I have verified the two axioms of problem one is Q of S is 1 and the other one is, the other one is that Q is countably additive function. That means these two things will imply that Q is a probability. So, these random variables will sort of go across and give you functions in the other spaces. This is an important result I wanted I think it is simple to show, but one must keep track. So, one can shift functions across sample spaces and the probabilities go across as well so it is an interesting idea.

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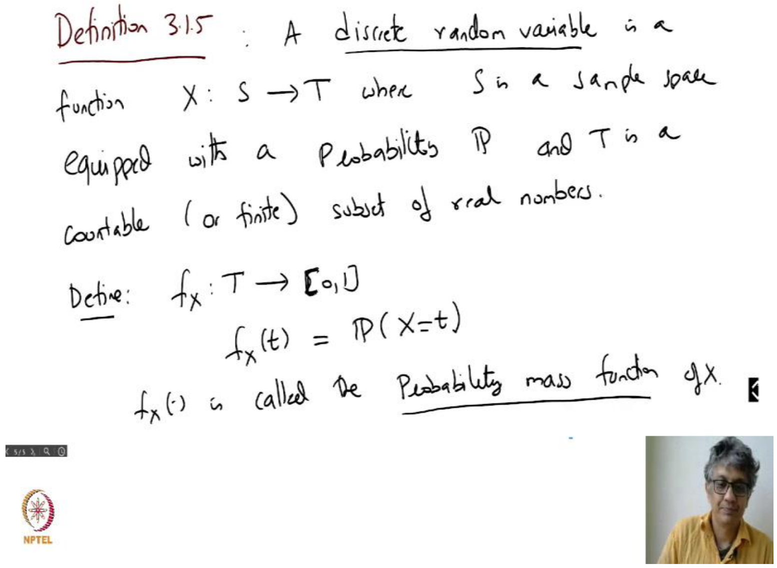

These define a certain concept. So, let me write the definition down, the definition down so definition 3.1.5. So, here what we will do, is the following. We will say a discrete random variable, a discrete random variable is. So, this is the definition, is a function X from S to T where S in the sample space equipped with probability P, with the probability P, probability P and T is a countable space. So, always my S is countable and T is countable or finite subset of real numbers. So, I will only look at random variables that go to real numbers

These are called discrete random variables and we already know that X would distribute probabilities across T by the function Q from the previous theorem. But what we will do is we will define a function f sub x from the T to 0, 1. 0, 1 given by f sub x of t is the chance that X is equal to t. And so then define f sub x equal to, then f sub x is called the probability mass function, mass function, mass function of X.

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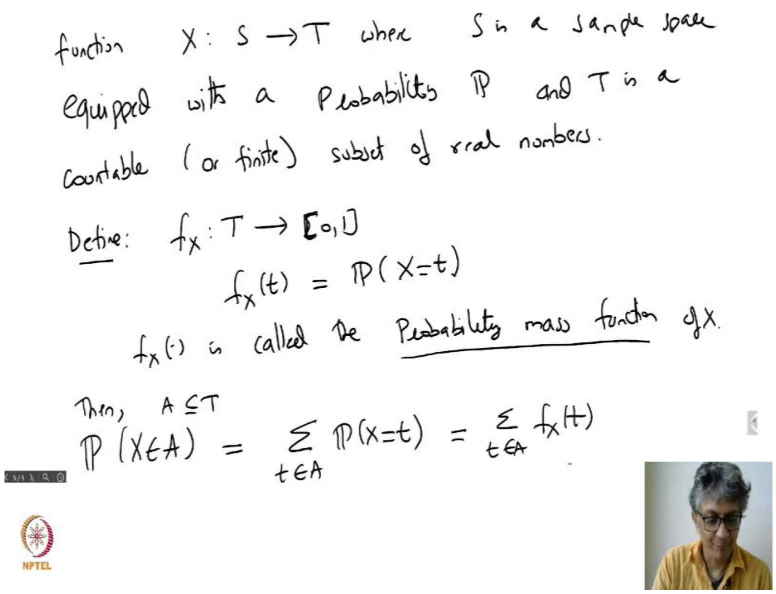

Then what you can do is you can do several things. You can do with this little f and capital X. Those two notations are introduced. So, the notation is the following. So, if you want to count chance of X is an A, then X is an A, that is the same as you can think of it as the sum or sum over t in A, for which probability X is equal to t, because that is how you can use the axiom probability of P to do this or you can think of it as the sum over t in A, f sub x of t.

So, you can use the function f sub x to do this. So, A is an event in t. So A is subset of T, then A is subset of T (())(13:30). Then in A is subset of T. So, you can use f sub x to get the answer. So these are two aspects of a random variable that we use again, again. We will identify the probability mass function and then get the answer. That is one aspect of random variables .

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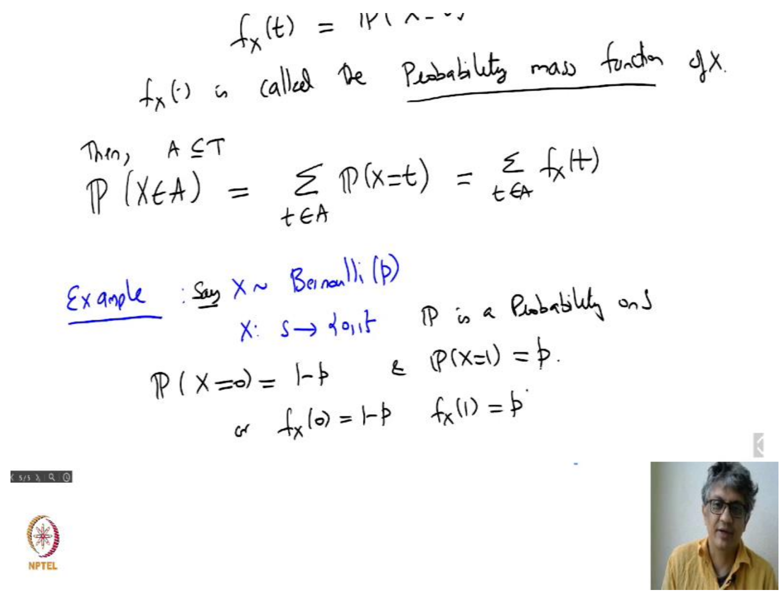

Let me do a simple example. Let us go to simple example. Let us say. So, again when you describe a random variable, we only want to describe, we do not worry about the sample space. We only worry about what values random variable takes. So, that tells you exactly everything random variables. So, let me do an example. So let us say, we would say x is a uniform. I will just say X is. Let us say Bernoulli P if the following happens. So, there is some sample space S from which X is from S to 0,1.

And what you will do is, the sample spaces you are not worried about and P is a probability on S, on S and then the chance that X is equal to 0 is going to be equal to 1 minus p and the chance that X equal to 1 is going to be equal to 1 equal to. This is what I would call my but Bernolli random variable .

So, we will say, X is this, if there is samples X, something sample space S to 0, 1. P some probability on S, but the distribution of X is given by this. So, that you could write in short form that f x at 0 is 1 minus p or in other words f x at 1 is just p. So, the distribution of X is specified by the range it takes and the probabilities of how it distributes it on the range. That is called the distribution of X.

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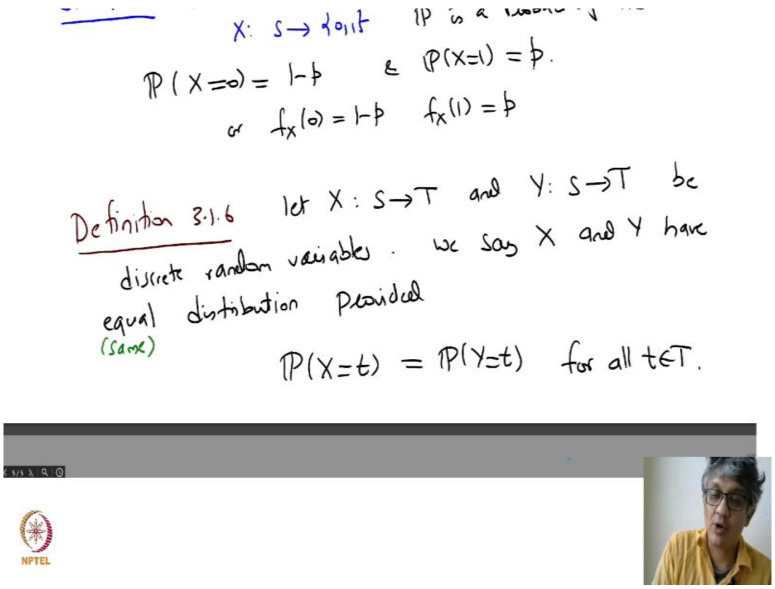

And then let me do one, so how do you compare two random variables. Let me just discuss that a little bit. So, now here is the definition 3.1.6. Let X be a function from S to T and Y also be a function from S to T. So, you have two functions from S to T and they are both discrete random variables. We say X and Y have equal distribution provided the following happen.

That is sorry provided that they distribute the probabilities the same way. Probability of X equal to t is the same as probability of Y to t for all (())(17:35). So, the probability is the same you say X and Y have the same distribution. I should just repeat this equal distribution, not distributed. Sometimes you do the same equal sometimes you use the word same they all mean the same thing. So, that means you identify every random variable with the distribution, it provides on the real line.

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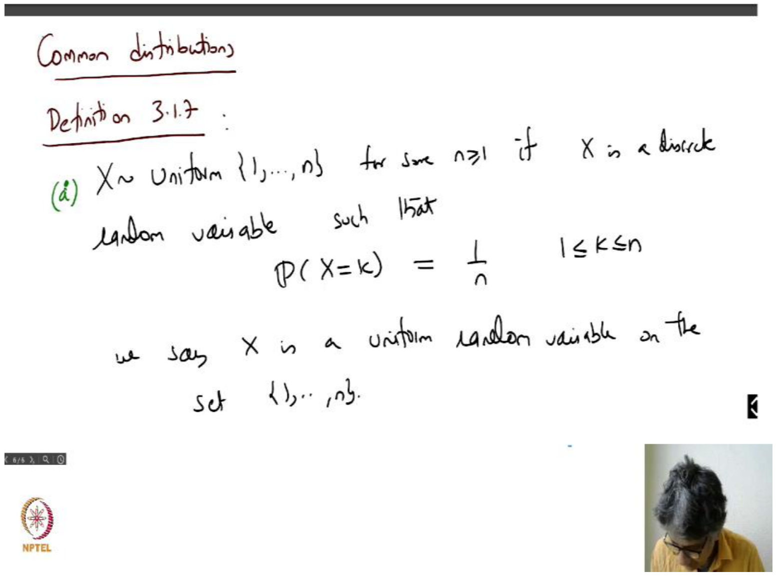

So, now let me just list out some common distribution. So, let us list out some common distributions. So, here is the definition, so I will define random variables of this form. So, I will define two things we have already seen before. So, I will say X is a random, is uniform 1 up to n for some n given equal to 1, if the following happens. If X is a random variable, X is a discrete random variable such that the probability of X equal to k is the same as 1 over n, for all k less than equal to n. And we will say, we say X is a uniform random variable, on the set 1 upto n.

So, just notice, I have, I have really not worried about what omega what S is, what the events are, what P is. All I am concerned is how X distributes its, its values how distributes the probabilities on its range. That is the only thing I am worried about. That is that signifies everything I want to know about X. So let us call this a and let me remove the dot.

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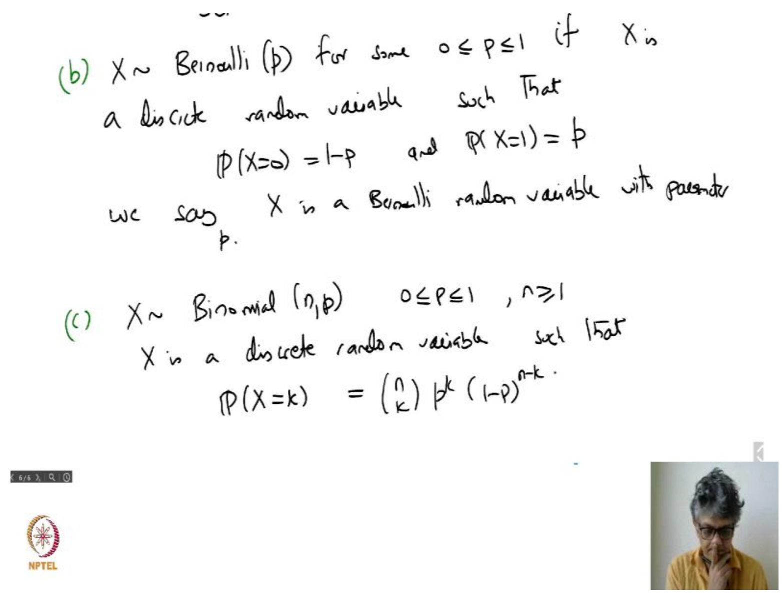

So, then let me do b let me do another one b which we have already seen also. Like I discussed before. I will write it properly now. X, I will say X is Bernoulli. I will take X as Bernoulli p, for some p between 0 and 1, if the following happens. If x again in the discrete random variable, such that the probability of X equal to 0 is 1minus p and the probability of X equal to 1 is 1, is p.

And we say that, we say that, let me go down a little bit, that is clear the two things are clear. So we say X is a Bernoulli random variable, Bernoulli random variable with parameters. So, the content, so you should go back and see that I have put, I have just forgotten about sample spaces. I have just said, I have a random variable. I specify the probabilities it takes. So, with that it comes the range comes and how the random variable distributes the probabilities on the range.

And that is enough for me to do calculations. I do not worry about S and what S does and so on so forth. So, now we can do the same thing. We can discuss binomial. We can do, let me write all these things down it is clear to all of us at the same page. So, I will write down, I would not write down everything but I will. So, I can say X is binomial n, p; if again you have p between 0 and 1 and n is bigger than equal to 1.

X is a discrete random variable, such that the chance that X is equal to k, is going to be n choose k, n choose k I will write in the notations here, n choose k, p power k 1 minus p to the power, n minus k. So, that is something you can define (())(23.13) in this fashion.

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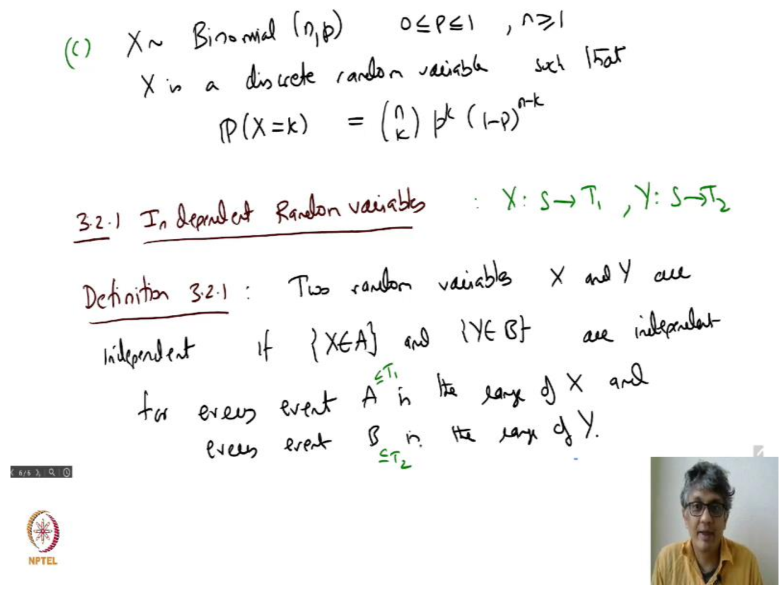

So now the, so now what we want to do is, I want to work with random variables. So, I have to also understand, how other cons they discuss about independence and stuff like that, go across to random variables. So, that we will do next. So, let me do the other concept called independent random variables. So this is 3.2.1. So, here what happens is the following so, let me here, I have to translate independence from events random variables.

So, here is the first definition of the following. Definition 3.2.1. So two random variables X and Y are independent. So, there are two functions now. So, when do you say the independent. So what you do is you look at all possible combinations. You look at independent, if the following; if X in A the event X in A and the event Y in B are independent. Independent for every event. So this is very important for every event.

Every event A in the range of X and every event B in the range of Y. So, just the notation here a little harrier. So, X goes from let us say X2. So, the way you think about this is that can go from let us say S to something 1. Y can go from S to something 2. They all are from the same sample space, but they are independent, if this happens. That is if X in A and Y in B are independent for everybody in the range of X and everybody in the range of Y. So, A is subset of T1 and B is subset of T2.

Just keep in mind that I have sort of, S sort of moved the, the notion of independence from events to random variables by getting corresponding events that specify the random variables. So when you when you do these things you have to be a little careful. You know random variables you must just check what happens. So, you cannot do it for. So, one simple remark after we will come back and do this next time, a little bit more in detail is the following that, that so what does this say. That is you want to say that is so that is so equivalently.

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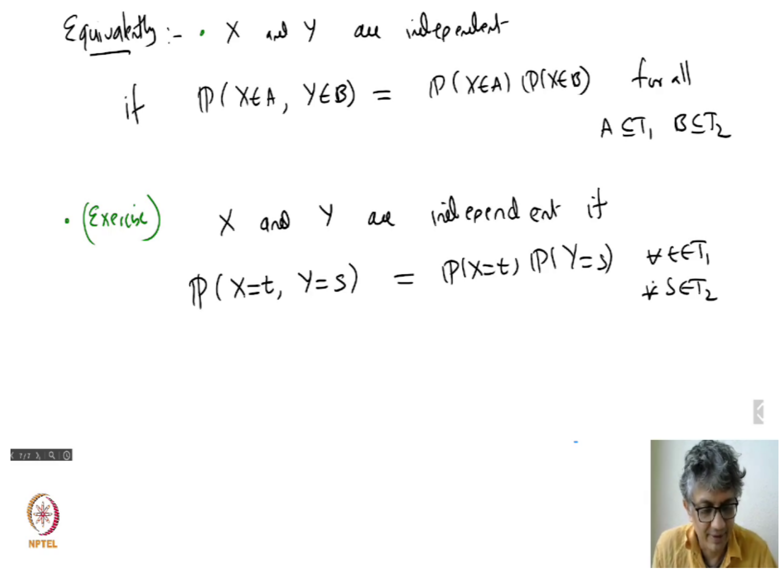

Let me do another slide. So, equivalently X and Y are independent, if what happens. So the first thing, we know is that the chance of X in this event and the chance of Y being that is the second event; that is the same as chance of X in A and a chance of X in B for all A subset of T1 and B subset of T2. So, that is how you, that is how you, you define independence (())(27.38)

But what we can do is you do not have to do this all the time. So, we have this tool, we already seen probabilities are defined by probability every outcome. So, this is an exercise one can show which I will not show in class but you can easily check; is that this equivalent to the fact the following kind X and Y are independent. You do not have to check it for everybody, every event If independent if probability of X equal to t and Y equal to let us say S is the same as chance that X equal to t times the chance that X, equal Y equal to s (())(28.28). So, enough to check for these particular events for all T and T1 and all S and T2. Enough to check in for these specific events. You know how to check it for everybody. So, this we will try and come and make it rigorous next time.