Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Ordinary Differential Equations
Lecture - 81
Convolution of functions

So, we have been discussing ordinary differential equations, specifically we have been looking at Laplace transforms and how with the aid of Laplace transforms we can solve ODEs.

So, it is in this direction we are proceeding and so, there is a key operation involving functions which we want to discuss in this lecture, which will help us you know calculate Laplace transforms in an effective way and that is the idea of Convolution of functions right, which is a subject matter for this lecture ok.

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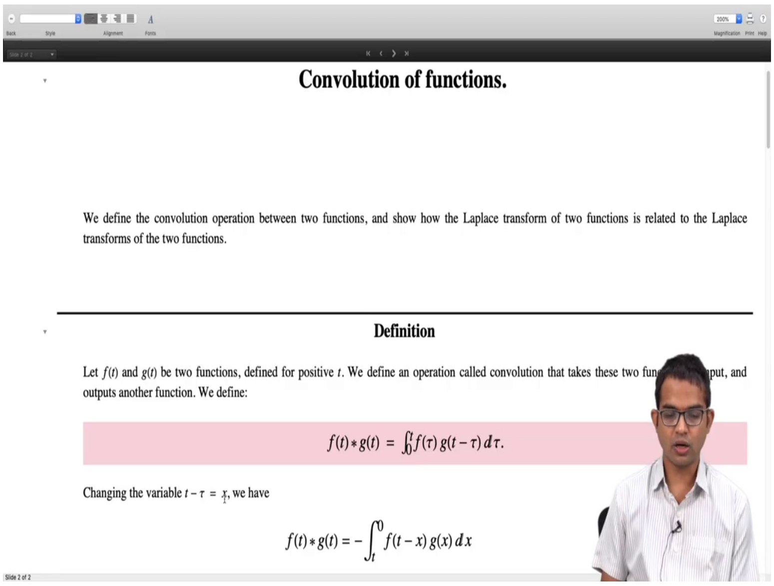

So, you know the reason why we are interested in the convolution operation which looks like a convoluted operation, which we will define in a moment is that you know what looks complicated in one domain turns out to be fairly simple in the other domain.

Namely, when you take the Laplace transform of a convolution of two functions, it is going to turn out to be very simple as we will see so, but let us start with the definition of this convolution. So, if you take two functions f of t and g of t well I mean for the purpose of Laplace transforms we are looking at functions which are defined for positive t right.

But we will see later on how the idea of convolution can be extended to functions, which are defined from minus infinity to plus infinity because you know the variable t where when it takes all values from minus infinity to plus infinity. So, the operation is the following. So, you take these two functions f of t and g of t and then you integrate.

So, you take the product of f of t times a shifted version of you know f of tau times the shifted version of this function g of t minus tau. It is like a correlation function in some sense, but it is a you have to shift on these two functions by an amount which is you know t minus tau right.

So, you take the value of the first function at tau and the other one is taken at g of t minus tau. You take the product of these two and then you integrate over all tau tau goes all the way from 0 to t right and so, if you do this operation. So, let us see what happens if you change this variable. So, it looks like it's apparently treating the function f you know differently from g right.

So, but if you change this variable t minus tau to x, we have f of t star g of t. The limits of integration now will go from t to 0 because it's x that we are integrating. So, if tau is 0, then x is t and if tau is t you get x is 0 and then you get a minus sign because d tau is minus d x and now in place of f of tau we have f of t minus x and in place of t minus tau of course, we have x. So, we have f of t minus x times g of x.

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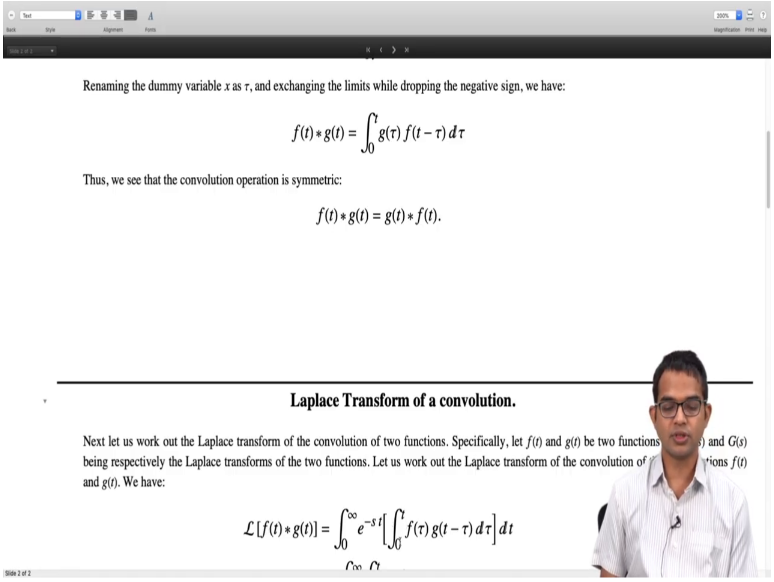

But you have a negative sign here, which you can observe into this integral you know if you change these limits, you know instead of it going from t to 0 if you make it 0 to t this negative sign goes and so, we see f of t star g of t is the same as integral 0 to t g of tau f of t minus tau d tau.

So, you see that you know this is you know as if you have reversed the role of f and t compared to this definition therefore, indeed f of t star g of t is the same as g of t star f of t its completely symmetric. So, there is no issue with this definition right. So, this is an operation which takes two functions f of t and g of t and gives you another function right. And so, why would we bother doing something so complicated?

So, the reason is that if you take the Laplace transform of such a function. So, let us see what happens if you take the Laplace transform of a convolution right. So, I have f of t and I have g of t. So, if I take the Laplace transform of the convolution. So, the convolution itself is defined like we just saw as this integral 0 to t, f of tau g of t minus tau d tau.

And then I have to put in this factor e to the minus st and then I integrate from 0 to infinity there is a dt which comes in at the end. So, if I do this integral. So, I get a double integral now. So, I have 0 to infinity and integral 0 to t e to the minus st f of tau g of t minus tau d tau dt right.

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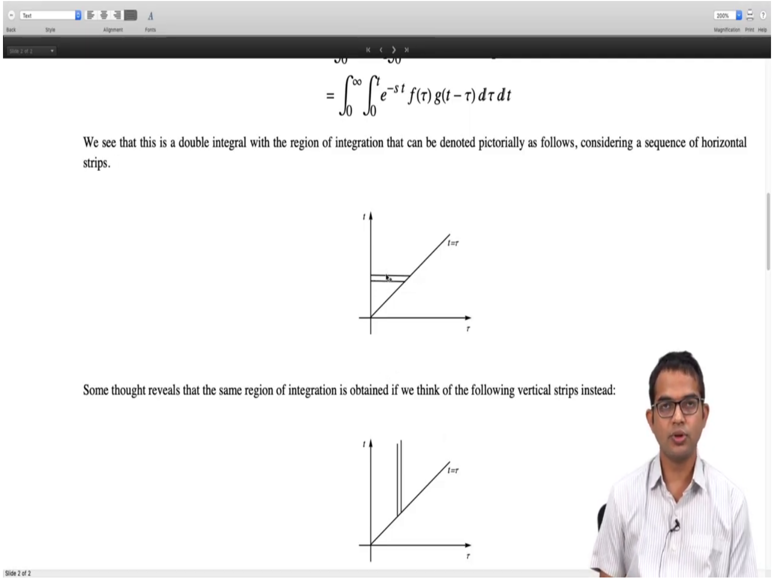

So, this is a double integral. So, let us look at this pictorially. So, it is helpful to you know see what is going on here. So, I have my tau and t I plot my tau along the x axis and I plot my t along the y axis. So, if I fix the value of if I so, first of all I. So, the way I am doing this double integral is I will take t to go all the way from 0 to infinity, but that happens in the second step.

So, if I fix my value of t to be some value. So, write like here and then I will take tau to go all the way from 0 to t and then I will allow t itself to go all the way from 0 to infinity, right. So, that is what is going on here. So, I have some function here which is you know some function of t and tau, right.

And then it is just the limits of this integration are as follows. First of all I fix t and take tau to go from 0 to t and then I allow t itself to go from 0 to infinity right. So, basically I am covering this entire region which I you know which lies above this line t equal to tau and to the right of the y axis this entire region is covered.

But, I mean the same region I could think of in a different way I could construct using you know strips, which are along you know vertical strips instead of horizontal strips. If I looked at horizontal strips I got you know this region of integration I could cover using these limits. But if I could also construct the same region, ultimately you are doing a double integral over some region right. So, instead of summing over all these horizontal strips I can also think of summing over these vertical strips.

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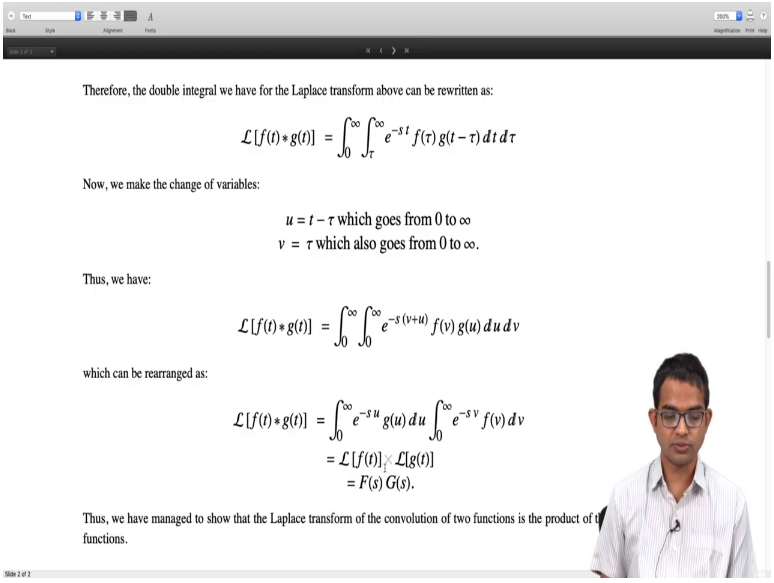

So, if I do this then I see that how would I do this? I would fix tau to be a certain value and allow t to go from tau all the way up to infinity right. So, I could. So, do this instead of. So, same integral double integral I could perform if I fix you know if I allow t to go from tau to infinity I fix tau and then tau itself goes from 0 to infinity, right. So, you just have to look at this picture and convince yourself that this is reasonable right.

So, look at you know I go to a particular point look at one strip. So, what have I done? I have fixed tau to be some value, you see tau itself can take all values from 0 to infinity, but if I fixed tau to be a certain value right I mean you are supposed to think of a strip tau to tau plus d tau.

Within this strip I see that t cannot take all values t will take only values from tau or you know after all you are going to cover this the same region as here right. And in order to do this I will restrict t to go from tau all the way up to infinity right. So, why do I bother doing all this? So, you will see in a moment. So, now, it helps me with this change of variables.

If I change my variables and I allow u; I make u equal to t minus tau, which will go from 0 to infinity and v equal to tau which also goes from 0 to infinity, right. So, you see that you know t minus tau is this entire region above you know after t goes from tau to infinity.

So, t minus tau will go from 0 to infinity and v is just tau after all you see tau is just going from 0 to infinity. So, the limits of u and v are easy to fix if you do it this way and then you have in place of t I have to put u plus tau, which is the same as u plus v. So, I have e to the minus s times v plus u and in place of f of tau I have f of v and in place of g of t minus tau I have g of u right.

So, you can work out the Jacobian and that is going to be just unity. So, you just get du and dv in place of dt d tau will go to du dv. So, we have converted one double integral to another double integral involving some different variables and why did we bother to do this?

It is because now we have both u and v being treated the same way basically. So, now, in fact, you can separate out these two and write it as two integrals. It is the product of two integrals, two single integrals, the double integral has been decoupled in some sense -using this change of variable.

So, I have 0 to infinity e to the minus su g of u du times 0 to infinity e to the minus sv f of v dv. So, which is nothing, but the Laplace transform individually of each of these functions right. So, this is the Laplace transform of f and this is the Laplace transform of g. So, it does not matter which order I put it.

So, the key point is that I have shown that the Laplace transform of the convolution of two functions is the same as the product of the Laplace transforms right. So, what is a complicated operation in the time domain is just simply a product operation in the Laplace transform space and that is where the power of this method comes from, right.

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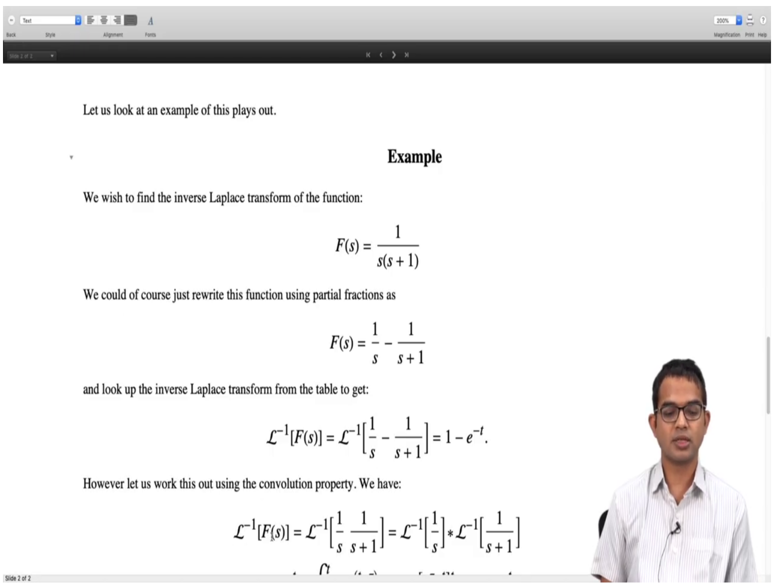

So, let us look at an example of how this plays out. So, suppose we wish to find the inverse Laplace transform of this function F of s is equal to 1 over s times s plus 1. So, this is something that we know how to do right. We could just rewrite this function as one over s minus 1 over s plus 1 fraction expansion and then we know how to take the inverse Laplace transform of 1 over s we know how to take the inverse Laplace transform 1 over s plus 1.

So, the answer is simply given by 1 minus e to the minus t right. So, as you can verify by taking the Laplace transform of this function, you will get back the same function that we started with, but let us work it out using the convolution property. So, what is this we have the inverse Laplace transform of F of s is the inverse Laplace transform of the product of these two functions.

But, we have seen that you know; what is a product in this Laplace transform of space becomes a convolution in the time space. So, that is inverse Laplace transform 1 over s convolution inverse Laplace transform of 1 over s plus 1 each of which we know how to do. So, it is just function 1 and when I say put 1 here it is the function f of t is equal to 1 convoluted with e to the minus t.

And this is nothing, but integral 0 to t, 1 times e to the minus t minus tau d tau which is simply e to the minus tau minus t right. So, it is an integral over tau. So, tau minus t 0 to t and immediately we get if I put tau is equal to t I get 1 and if I put 0 I get minus t. So, I have indeed 1 minus e to the minus t which is exactly the same result we obtained directly earlier, right.

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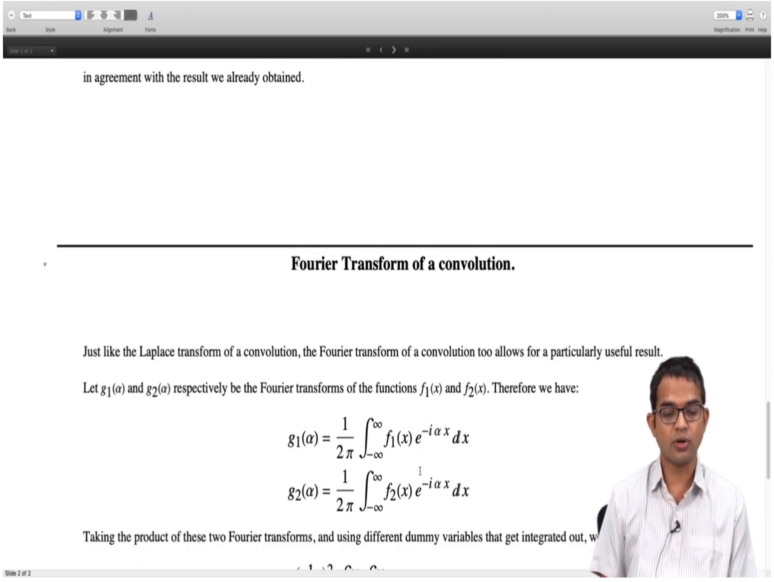

So, this is a very simple example where we had an alternate, even easier method to evaluate the inverse Laplace transform by, but there are other places where in fact, this convolution you know operation is a useful way of computing the inverse Laplace transform. So, definitely this is an important property to be familiar with.

So, let us look at also the idea of the Fourier transform of a convolution right. So, just like the Laplace transform of a convolution has this very nice convenient form, we can also look at the Fourier transform of a convolution and we will see that there is something you know simple which emerges there as well.

So, we have two functions f 1 of x and f 2 of x. So, now, it is the Fourier transform. So, you have to do an integral all the way from minus infinity to plus infinity, we have this factor 1 over 2 pi right because that is the convention we are following. So, we have 1 over 2 pi integral minus infinity to plus infinity, f 1 of x e to the minus i alpha x dx and g 2 of alpha is 1 over 2 pi integral minus infinity to plus infinity f 2 of x e to the minus i alpha x dx right.

So, if you take the product of these two functions the Fourier transforms then we have g 1 of alpha times g 2 of alpha is equal to 1 over 2 pi the whole squared integral minus infinity to plus infinity again integral minus infinity to plus infinity. So, we will use some other dummy variables here, after all these are getting integrated out if you put the same x you will run into difficulties. So, we will say du dv f 1 of u f 2 of u times e to the minus i alpha times u plus v.

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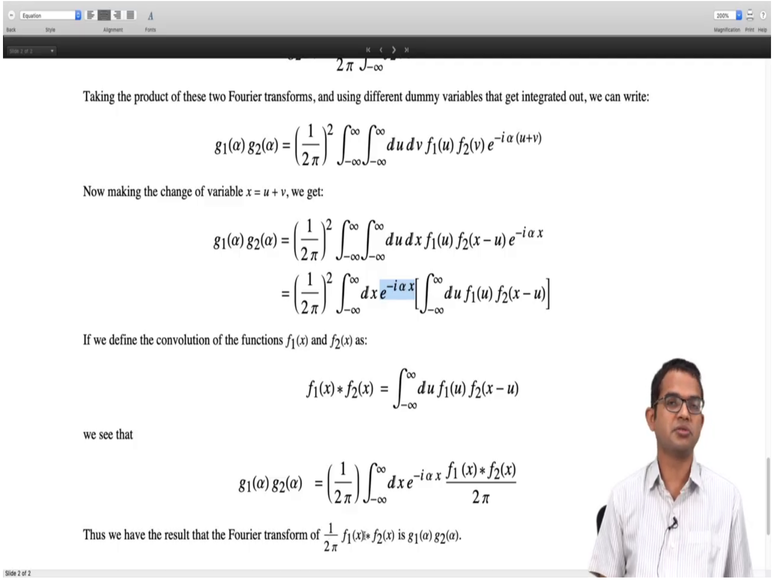

Now, we make the change of variable x is equal to u plus v. So, we have integral you know 1 over 2 pi the whole square integral minus infinity to plus infinity integral minus infinity to plus infinity du dx f 1 of u and in place of f 2 of v, we put f 2 of x minus u and we have e to the minus i alpha x.

So, then we see that in fact, we can write this as 1 over 2 pi the whole square integral minus infinity to plus infinity dxe to the minus i alpha x we can pull out this factor and then we have this integral minus infinity to plus infinity du f 1 of u times f 2 of x minus u. So, if we look at this integral which is given in the square brackets, it is something similar to what we have seen.

So in fact, we can define this as the convolution of two functions which are defined all the way from minus infinity to plus infinity. So, it is a full integral minus infinity to plus infinity du f 1 of u times f 2 of x minus u. So, this is also like you know taking a function taking the other function you know you have to you know shift it, but also there is a minus sign that you have to take and shift it by an amount 2 and then overlap the 2 and then integrate, right.

So, that is what this operation is. It is like a correlation, but with some shift involved and if you do this. So, then we see that in fact, the product of these two Fourier transforms is nothing, but you know this is like taking the Fourier transform of this convolution, but you have to be careful there is also a factor of 1 by 2 pi involved.

So, what we have managed to show is, the Fourier transform of 1 by 2 pi times the convolution of these two functions f 1 of x star f of f 2 of x you know defined in this manner is simply the product of the Fourier transforms g 1 of alpha times g 2 of alpha, right.

So, this factor 1 over 2 pi you know appears in this manner because of the convention we have used for our Fourier transform definition and the inverse Fourier transform so on right. If you had used a slightly different convention we would have you know different factors, but the key point is that what is a convolution in one domain becomes a product in the other domain you know just like we saw for Laplace transform.

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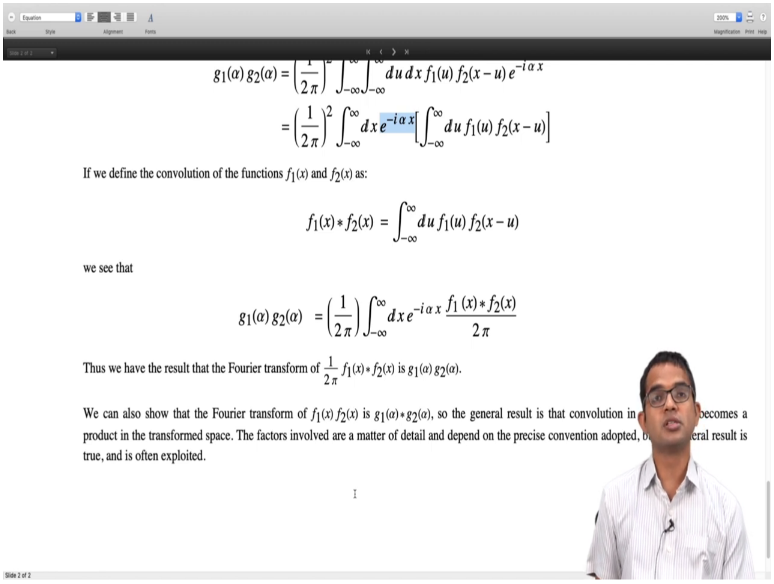

So, indeed here it turns out that if you take the product you know with this within this convention, if you take the product of these two functions f 1 of x times f 2 of x and take its Fourier transform its going to be just the convolution g 1 of alpha convoluted with g 2 of alpha.

So, this is something that we will allow you to verify yourself. But the key idea is you know what is a convolution in one space will become a product in the other. And that is a property which you know which often we try to exploit because products are easy to work with and convolution is a much more complicated operation ok. So, that is all for this lecture.

Thank you.