Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 75
Resonance

Ok. So, we have built a fair amount of machinery with Ordinary Differential Equations. We have seen how to apply some of this with vibrations in mechanical systems. Specifically, we looked at the case where there is no damping the simplest harmonic oscillator problem. And then we looked at the damped harmonic oscillator problem which gave us three different possibilities, namely the under-damped you know oscillatory solution, or critically damped, or over-damped solutions where oscillations are not possible.

And then we saw what happens if an external drive is present right. So, you know the specific case of sinusoidal driving was instructive, so we saw that if you periodically push the system for long times, it is really the external drive which is going to dominate the nature of the dynamics of the system.

So, the position of the system is going to in fact try to follow the external force very very closely for long times. For short times of course, the solution from the homogeneous equation also matters and this is what is called the transient part of the solution. And for long times, it is the particular solution which dominates, and this is called the steady state solution.

Because that is where the system eventually settles down to a steady state at long times. Now, we saw that the nature of the solution is such that its frequency, the frequency of the position is going to be the same as it is going to have periodic motion and for long times.

And this periodic motion has the same frequency as the frequency of the external drive. And there is a phase difference between the motion of the external drive and of the particle. And that phase difference is you know we can work out tan phi in terms of the various parameters of the problem.

So, in this lecture we will see what happens to your system when you drive it at certain special frequencies right. So, you know since we are subjecting your system to a drive in the lab, we have full control over the amplitude or of the external drive.

We have full control over the nature of the drives that we choose. So, if we choose a cosine one, you know there are two parameters which we have the knob for in our hand; one is the amplitude and the other is frequency right. So, the particular emphasis of this lecture will be to see that there is something very special that happens if you drive your system at a special frequency which is known as the resonant frequency.

And so the phenomenon of resonance you know has applications in all kinds of fields. But in our lecture here, we will try to use the methods of you know differential equations that we have developed over many lectures to study resonance in this kind of mechanical system with, in which vibrations are present and an external drive is introduced ok.

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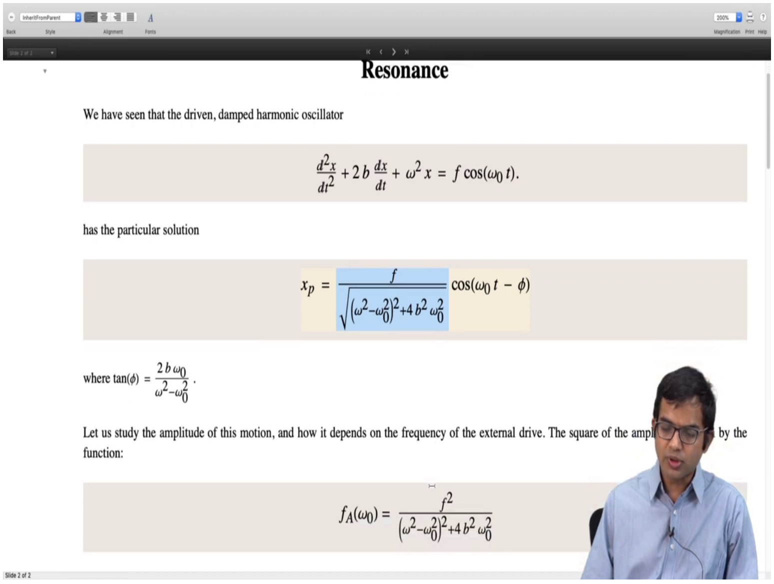

So, the differential equation of interest here is d squared x by dt squared plus 2 b dx by dt plus omega squared x is equal to f times cosine omega naught t. And we have seen that the solution, the particular solution right or the steady state solution for this problem is f divided by the square root of this stuff. This is the amplitude times cosine of omega naught t minus y. So, the key point is that it is the same omega naught which appears you know as in the you know in the drive here right.

And then there is a, this expression for tan of phi which you can work on right. So, now, the focus here is on this amplitude itself right. So, we have already seen that the frequency is the same as the frequency of the external drive.

So, if you look at the amplitude, let us study the square of the amplitude. So, I am going to call this f A of omega naught as f squared divided by f omega naught omega squared minus omega naught squared the whole squared plus 4 b squared omega naught squared.

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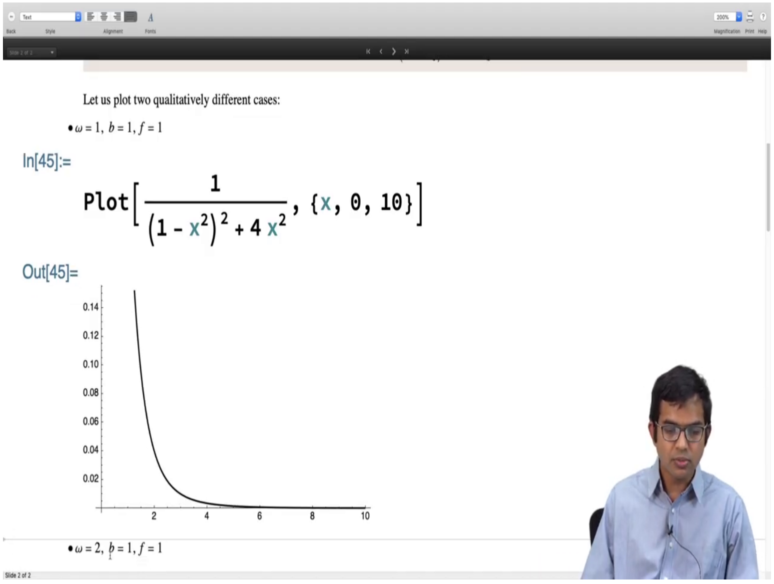

So, if I study this quantity; the square of the amplitude. So then I see that you know there are two qualitatively different cases which come about if I put omega equals 1, b equal to 1, and f equals 1, so now I see if I plot it, it is just a monotonically decreasing function as a function of omega naught right. So, it is like a featureless sort of decay with a function of omega naught.

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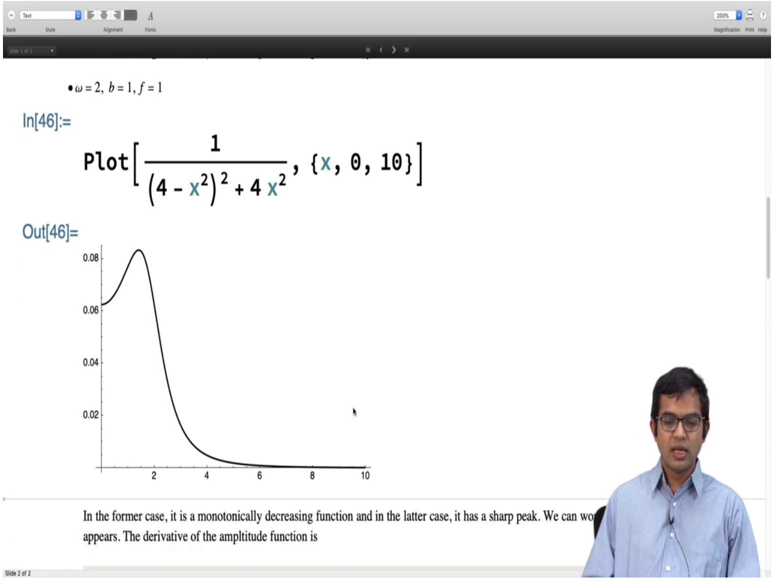

But, on the other hand, if I choose omega to be 2, b to be 1, and f equal to 1, I get you know something more interesting. So, look at this plot. And so now, you see that there is an increase in this amplitude, and there is a peak right. So, there is a special value of this omega naught at which the amplitude attains a peak. And then again if you keep on increasing omega naught beyond that point, it is going to decay right.

And for very large omega naught, so very large frequencies are you know it is as if you are not you are hardly doing anything to the system right. So, think of what omega naught going out going to infinity means. It means that you are shaking the system very, very rapidly. So, when you have this very very high frequency sort of drive, basically it does nothing to the system; there is no time for the system to, you know, get affected by it.

So, on the other hand, if you drive it too slowly again the effect is not so optimal, but there is a very special in-between frequency at which you can get very high amplitudes right. So, that is what is called resonance right. Let us understand this, we can work this out analytically how this appears right.

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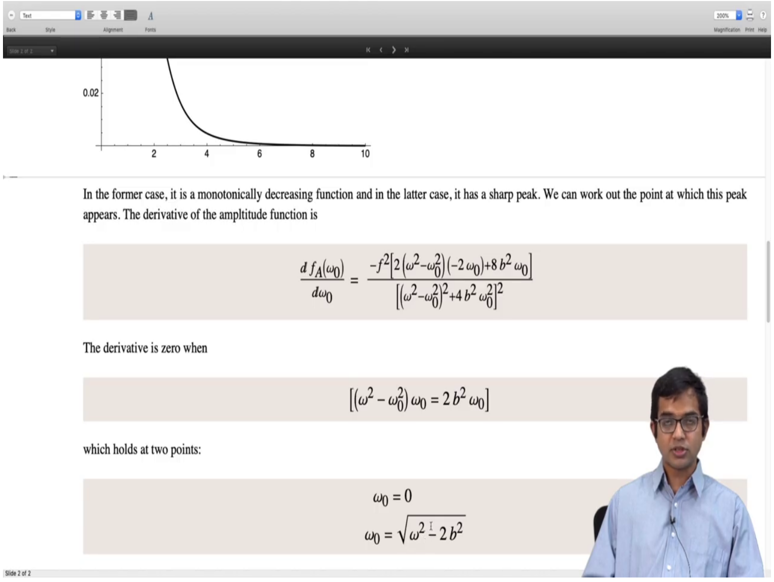

So, you take the derivative of this f A function and with respect to omega naught, and then you know if you put it equal to 0, then you see that it goes to 0 if omega squared minus omega naught squared a whole thing times omega naught is equal 2 b squared omega naught.

Now, this is possible in two cases. One is if omega naught is 0 right, so omega naught is 0 you see this is actually an extremum point and it is so here also. Well, it is not seen here, but it appears omega naught is 0 is a, is an extremum point. But you can also have another second extremum point which appears when omega naught is equal to square root of omega squared at omega naught is equal to squared root omega squared minus 2 b squared.

So, now some thought reveals why we did not see a peak in the first case right. So, we had chosen omega to be too small here. It is only if your omega is greater than if omega is greater than square root of 2 b square root of 2 times b only then will you get a real omega naught which is you know for which you have an extremum point, and that is the that is going to be a maximum in amplitude right.

So, the condition is that if omega must be first of all omega squared must be greater than 2 b squared, and then omega naught if you can choose your omega naught to be at squared root omega squared minus 2 b squared, then you get a resonance right. So, this frequency square root omega squared minus 2 b squared is called the resonant frequency or the resonance frequency. And if you are driving your system at this frequency, you are called resonant driving. And it will result in very high amplitudes and this phenomenon is called resonance right.

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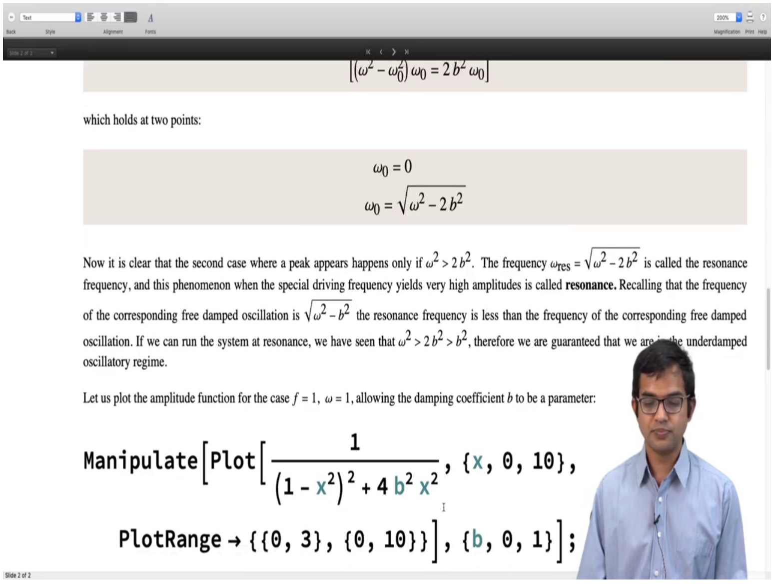

So, recalling that the frequency of the corresponding free damped oscillation is, suppose you did not have an external drive right. So, then we have seen how damped oscillation happens. You know you if you remember that we call this something called beta which is the square root of omega squared minus b squared.

So, then we see that in fact the resonance frequency is less than this frequency of this frequency beta right. So, if you are going to drive if you want resonance, then for sure your system must be under-damped because if you are driving at resonance you must have omega squared minus 2 b squared must be greater than 0.

So, omega squared is greater than 2 b squared which in turn is greater than b squared. So, omega squared is greater than b squared which was the condition for you know driving your system for your for a under damped oscillatory regime right. So, the point is that you are in the under-damped oscillator regime.

And then you are driving your system at this resonant frequency, then you can have resonance. So, let us look at a plot of the amplitude function once again. We put f equal to 1 and omega equal to 1, and look at you know varying values of the parameter beta b b the damping coefficient.

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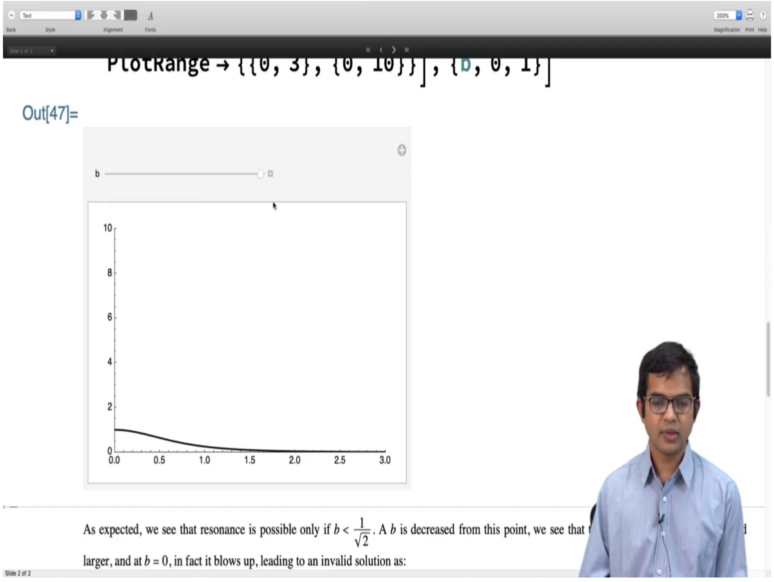

So, if we look at this, let us look at this series of plots. So, we see that, if b is very tiny right, then you get a big peak. And then there is this you know as you keep on increasing b, the height of this peak shrinks. And there comes a value of b beyond which you know there is no possibility of resonance.

So, in fact, it goes to a scenario where there is no peak anymore right. So, the key message from this plot is that as you keep on decreasing damping, there comes a point right. So, as you decrease damping the magnitude of the peak itself keeps on increasing. And so when b goes to 0, in fact, the damping becomes infinite right. So, you get this invalid solution f A.

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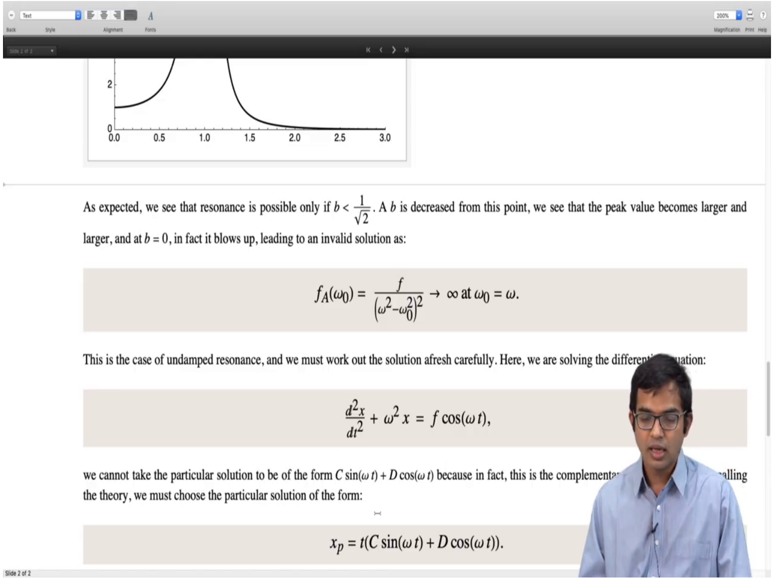

If you put b equals 0, then f A of omega naught will just go to f by omega squared minus omega naught squared which is basically infinity at omega naught equal to omega right. So, which is an invalid solution you can see that it is a singularity. So, in fact, this is the scenario of what is called undamped resonance right.

So, there is no damping in your system and you are driving it at resonance. So, this is a solution that we must go back and work this out afresh. Let us do it more carefully and afresh. So, what we are doing is we are really solving this differential equation d squared x by dt squared plus omega squared x is equal to f cosine omega t right.

So, on the right hand side, I have already put omega naught equal to omega right. I am driving my system at the same frequency as the natural frequency of the system which is the condition of resonance and there is no damping in this system. So, this is a differential equation for which I should be able to work out a solution.

So, this kind of nonsensical infinity should not arise right, so we will see how that can be resolved. So, the key point is that you cannot take C sin omega t plus D cosine omega t right, blindly as a particular solution. Because, in fact this is the complementary function and that is where the error is.

So, when we were doing it, earlier we had taken this kind of thing for the particular solution with omega naught; and now we have putting omega naught equal to omega and then there is a difficulty because the particular solution becomes the complementary function is in fact the same. The two of them when they become the same, in fact you have to get a linearly independent solution. And here you have to multiply by t as we have seen. So, we have to try out this form x p is equal to t times C sin omega t plus D cosine omega t.

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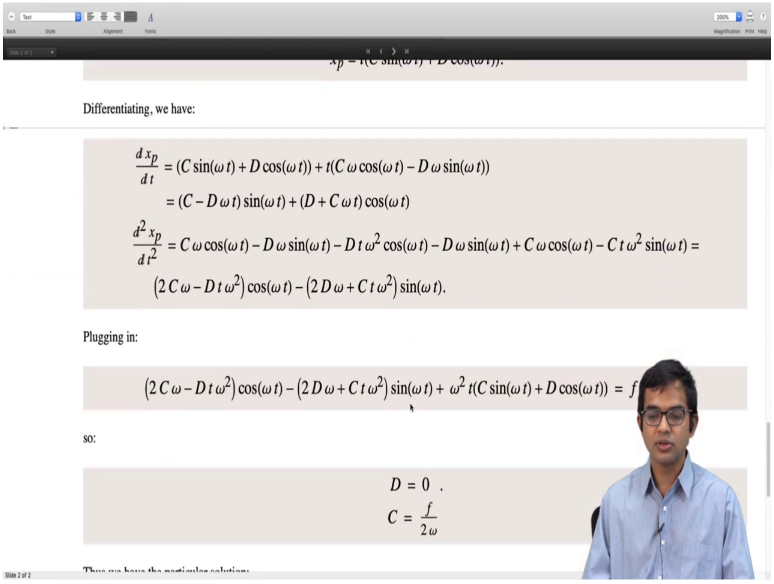

Then if you differentiate once, then you get you know all these sines and cosines and then there is a t times cosine, you have to collect all these terms carefully, and then you have to take a derivative again. So, this will result in 2 C omega minus D t omega squared times cosine omega t minus 2 d omega plus C t omega squared times sin omega t. You should check this algebra.

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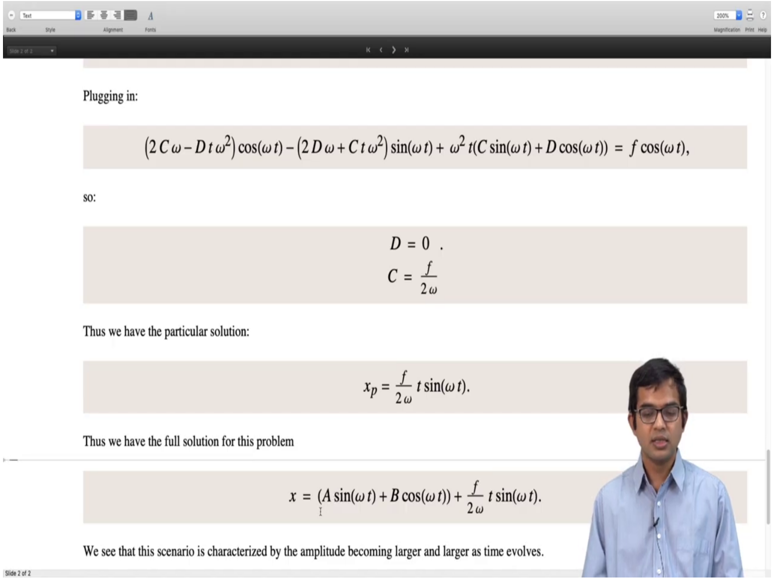

And then if we demand that this is a solution, this is a particular solution for this differential equation, we plug back you know this entire stuff into that original differential equation. And then we get these two conditions. So, D will go to 0, and C is equal to f of 2 by omega. So, if you choose your particular solution to be f over 2 omega times t times sin omega t is a valid particular solution of this differential equation as you can explicitly check.

So, the difficulty was we had not put in this factor of t. So, this factor of t is absolutely vital here. So, once you fix this, then the full solution for you know this undamped resonance phenomenon is just A sin omega t plus cosine B cosine omega t that is the standard complementary function plus f over 2 omega times t times sin omega t.

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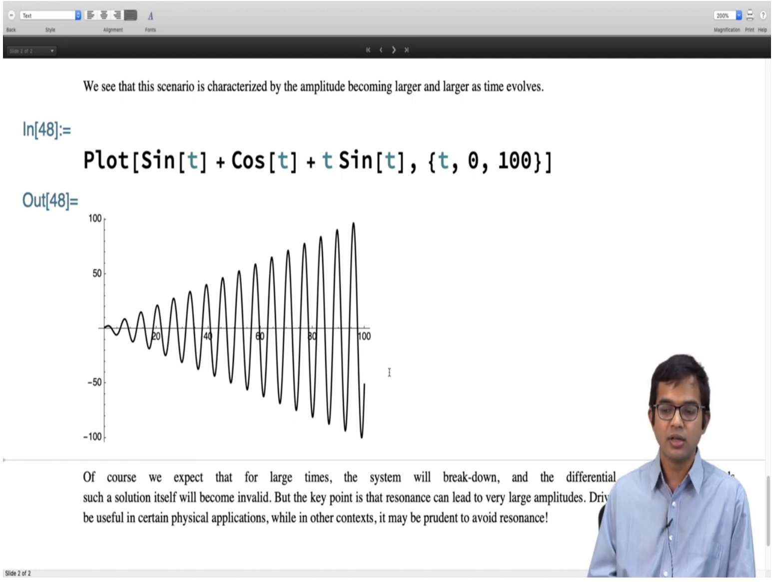

So, this t is absolutely crucial. And so this gives rise to an amplitude which keeps on increasing; it becomes larger and larger as time evolves. Let us plot this; so, you see now yeah. So, now, you see that this t is going to dominate entirely for large times, and it is going to become larger and larger and larger.

And so in fact, you will reach a scenario where you know this system itself will break down beyond the point. It is clearly unphysical for x to become actually arbitrary large right. But it can become very large but stay finite, but extremely large, and then some kind of breakdown will happen to the system.

And that is why resonance is a very powerful phenomenon which may be useful in certain contexts, but it may be extremely harmful in other contexts. So, one harmful scenario of this is you know when soldiers before marching on a bridge are often advised to stay out of step because if they were to march in step, they can be the external drive.

And if their external driving frequency happens to match the natural frequency of the bridge, then this can lead to catastrophic consequences. But there are other contexts where resonance is something that is desirable. You know efforts are made to drive the system at resonance right ok. So, that is all for this lecture we looked at resonance, anyhow.

Thank you.