Mathematical Methods 1
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Ordinary Differential Equations
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Ordinary Differential Equations

So, we have looked at how you know vibrations in mechanical systems can be modeled as a Differential Equation which we can solve using our machinery.

And so, we looked at these three different possibilities for a damped harmonic oscillator: how you can have under damp motion which leads to oscillatory motion or critically damped or overdamped cases which gives you non oscillatory decay solutions right. All of this we worked out using our machinery from differential equations.

So, here you will look at you know the next interesting scenario which comes in issue you also have an external force that you are applying to the system right.

So, in the theory of differential equations with thought of these as in homogeneous differential equations. Here it is, you know the physical picture is to think of them as an external drive being applied to a system. So, these are what are called driven harmonic oscillators if you wish that is the topic for this lecture ok.

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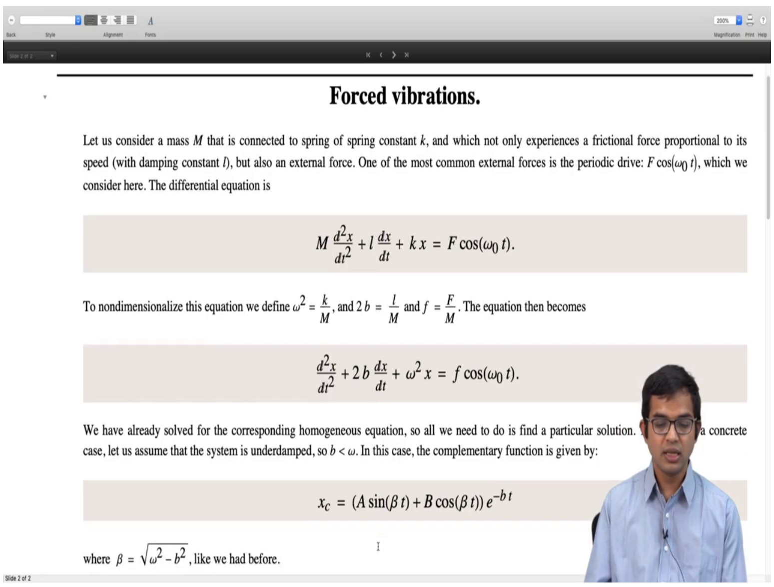

So, we have a mass M you know and its attached to a spring with spring constant k and there is the frictional force we you know look at a damping constant of l associated with its of with this frictional force which is proportional to the speed of the mass and there is an external force right.

So, the external force can be more general than you know what we consider here which is F times cosine of omega naught t. But this is completely you know this is a very standard type of external force. If you understand this, then it opens up our understanding of many other kinds of drives.

So, we look at periodic drive with frequency omega naught and amplitude f and so, the differential equation here is just M times d squared x by dt squared plus l times dx by dt plus k x is equal F times cosine of omega naught t right. So, it is useful to define these quantities: omega naught squared is equal to k by M 2 b equal to l by M like we did earlier and also introduce a small f is equal f by m.

So, our differential equation now becomes d squared x by dt squared plus 2 b times dx by dt plus omega squared x is equal f cosine of omega naught t right. So, you see that there is you know there is omega, there is also an omega naught right. We have a knob with you know as far as omega naught is concerned which is in our hands which we can control and so, we will see how you know the interplay of these different frequencies leads to interesting phenomena.

So, we have already solved for the homogeneous equation right. So, we just need to work out a particular solution here. So, let us say that we focus on the under-damped case right for concreteness right. I mean you can also work out the full general solution for the critically damped case or the overdamped cases, but you will see that basically it is the particular solution which is of particular importance or of interest here right.

So, we will argue how for long times the particular solution is what will dominate the dynamics of this system ok. So, the complementary function here is just A sin of beta t plus cosine of beta t the whole thing multiplied by e to the minus b t as we have already seen.

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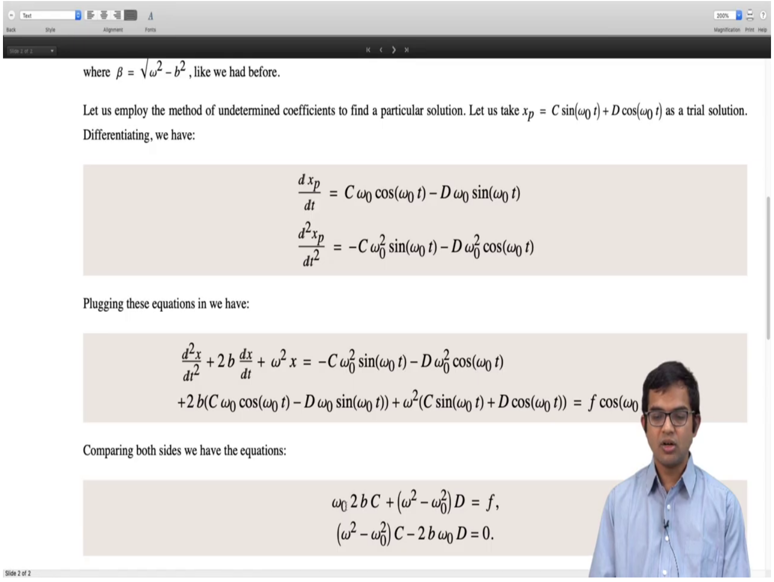

And so, we are of course, beta is defined as square root of b omega squared minus b squared right. So, this is a case where omega is greater than b. So, omega squared minus b squared t squared taking the square root of this quantity is not an issue ok. So, to find a particular solution, we look for a particular solution of this form, C times sin omega naught t plus d times cosine of omega naught t. This is a trial ansatz that we make right.

So, we have seen how the particular solution should be made to mimic the drive the external I you know externally applied force right. So, both cosine and sin you know they have to go together and pair. So, we have to consider both of these C sin of omega naught t plus d cosine of omega naught t.

And then if we differentiate this, then you have C omega naught cosine of omega naught t minus D omega naught sin of omega naught t a second round of differentiation and we get a minus C omega naught squared sin of omega naught t minus D omega naught squared cosine of omega naught t.

Now, we have to just plug these expressions back into the original differential equation. So, then we have d squared x by d t squared plus 2 b d x by d t plus omega squared x. So, we plug in for each of these quantities, we have here the second derivative, the first derivative and the ansatz itself which is here. So, if you plug in these expressions and then we must equate it to f cosine of omega t omega naught t.

So, basically we have to ensure that the coefficients are such that you know this is a method of undetermined coefficients you have to ensure our coefficients are such that you know omega naught 2 b C plus omega squared minus omega naught squared times D you know these this is the coefficient corresponding to cosine of omega naught t. So, on the right hand side, we see that this coefficient must equal f. And there is no coefficient corresponding to sin of omega naught t.

So, then we see that this will immediately imply that omega squared minus omega naught squared times C minus 2 b omega naught D must be equal to 0 right.

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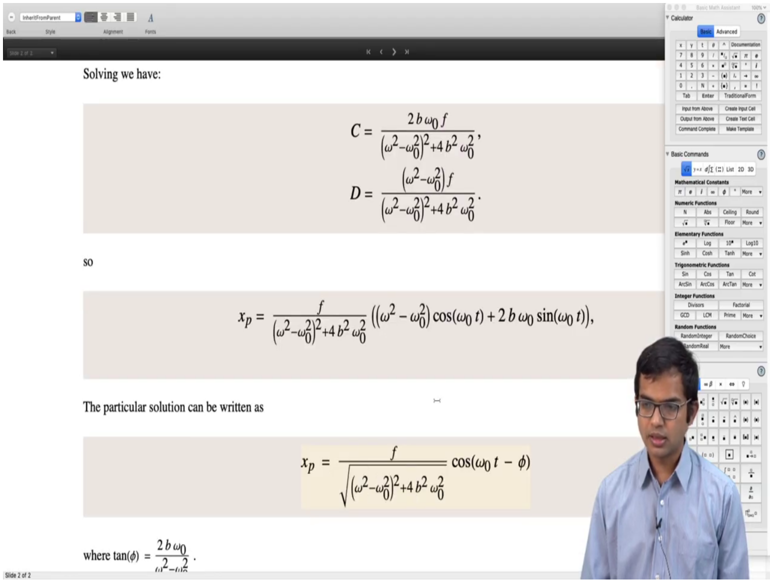

So, now it is a you know it is a couple linear equations two linear equations and two unknowns. So, we know how to solve this and so, you solve it you will get C is equal 2 b omega naught f divided by omega squared minus omega naught squared the whole squared plus 4 b squared omega naught squared and D is just omega squared minus omega naught squared the whole times f divided by omega squared minus omega naught squared the whole squared plus 4 b squared omega naught squared right

Once we have solved for c and d, we just plug these constants back into the particular solution and our particular solution is just f over omega squared minus omega naught squared the whole squared plus 4 b squared omega naught squared. Now, this entire thing multiplied by omega squared minus omega naught squared times cosine of omega t plus 2 b omega naught times sin of omega naught t, you should check this right.

So, it is you know fairly straightforward algebra is involved, but you should be careful and make sure that it is all correct. So, now, it is useful to rewrite this particular solution that we have obtained in a form which is instructive right. So, we can think of this as well, I mean there should be a square root here. So, there should be a square root square root here, which we can put this back in.

So, there is a square root here. So, f divided by this quantity ok so, that is so, f divided by square root of this quantity times cosine of omega naught t minus phi where tan phi.

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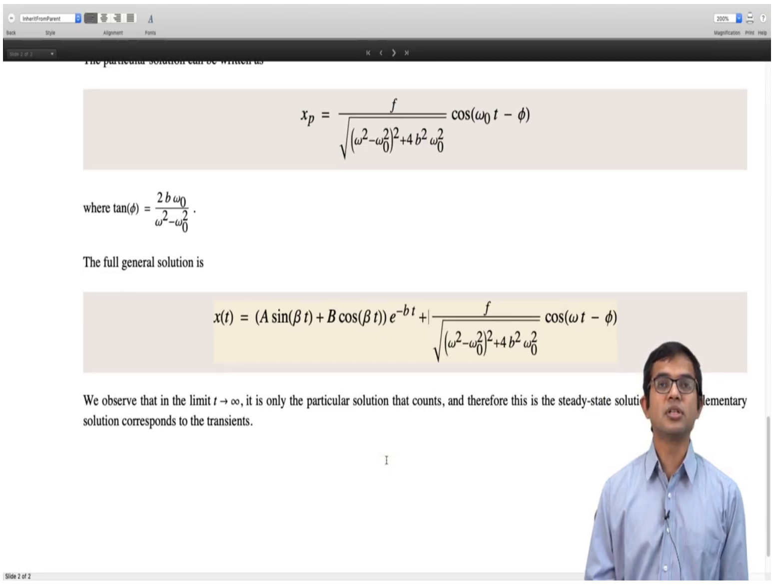

So, you observe that you have something like cosine of omega naught t cos phi plus sin omega naught t sin phi right. So, that is cosine of omega naught t minus phi and then you have to arrange these terms in such a way that you know that you figure out what phi is.

So, here you can check that indeed phi must be chosen such that tan phi is equal to 2 b omega naught divided by omega squared minus omega naught squared. So, what this tells us is that so, if you are driving your system at a frequency omega naught. So, the particular solution is also going to have the same frequency, but it has a frequency; there is a; there is a phase difference right.

So, the solution is out of phase by an amount of phi which is given by this quantity. So, the full general solution finally, we can write it as x of t is equal to the complementary function plus this particular solution right. So, once again so, we are missing this square root so, plug this in and put it in here ok. So, we observed that in the limit t tending to infinity. So, you see that you know this first part the complementary function becomes less and less relevant as more and more time elapses right.

So, it does not matter in which regime you are driving the system whether it is under-damped, overdamped or critically damped so, there is always going to be this kind of an exponentially decaying term. So, for large times basically this you know this complementary function is going to become irrelevant for you know as time tends to infinity.

So, that is the reason why this particular solution in this context is also known as the steady state solution right. So, your system for long times will settle down to the steady state and whatever information is contained in the complementary function is actually what is called transients right.

So, it is important only if you are interested in how the system behaves at very very small tiny times right after this external driving force is introduced and then basically the external drive takes over and then so, the system is entirely under the control of the external drive for long times.

And we see that the motion is going to be periodic with exactly the same frequency as the frequency of the external drive and the you know there is a certain phase difference which appears in between you know the motion of your particle and the externally applied drive right. So, that is what the solution tells us. That is all for this lecture.

Thank you.