Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 72
Variation of Parameters

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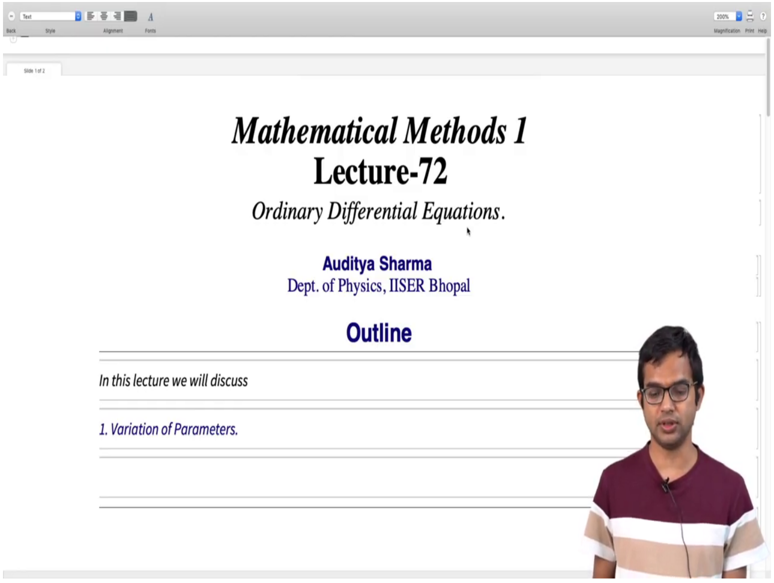

So, we have seen a bunch of techniques for second order differential equations. You know first we started with homogeneous, differential equations linear ones are the simplest and then using the general solution of the homogeneous differential equation we saw how we can work out the you know answer for an inhomogeneous differential equation provided we are able to find a particular solution for the inhomogeneous differential equation right.

So, if you can find one particular solution you just add it to the complementary function and you are done. Right often it so happens that even finding one particular solution is not so easy right. And so in this lecture, we will look at a general method for finding out a particular solution for second order differential equations.

If you know, the full general solution for the corresponding homogeneous differential equation. So, this is inspired by a method which we have already seen, but with first order differential equations right and it is quite general in its applicability and in fact, it can be extended to even higher order differential equations. But today in this lecture, we will look at second order differential equations alone.

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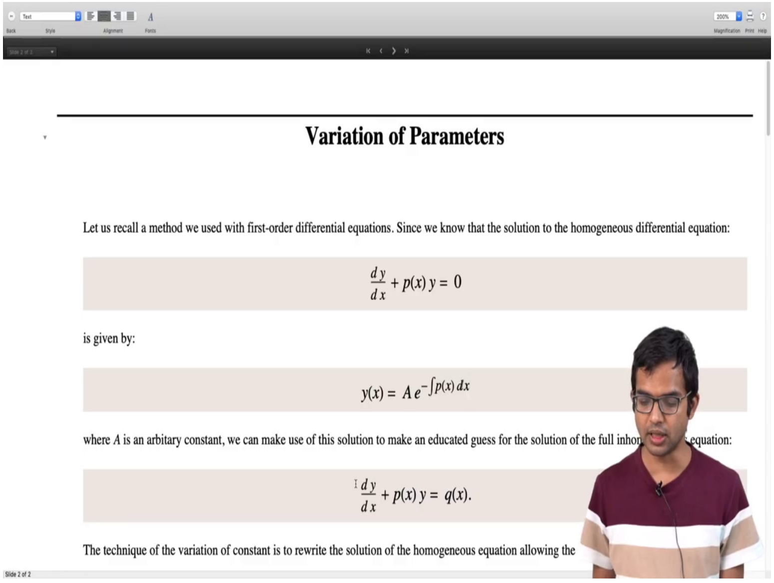

So let us recall you know what we did with first order differential equations. If somehow we are able to work out the homogeneous first order differential equation which we have seen is always possible right. So, it is in fact, you can write down the integrate in getting factor and then you are done right.

So, that is the advantage of a first order differential equation. So, no matter how complicated p of x is you know you will be able to at least formally write down this answer y of x is equal to A times e to the minus integral p of x dx right. It may or may not be possible to evaluate this integral in close form, but this solution is always something you can write down formally.

Now, we saw how when you want to work out the solution for the inhomogeneous differential equation it is first order so there is only one free constant. So, you elevate this constant to a function right. So, that is the educated guess which was suggested you know. So, you start with this differential equation.

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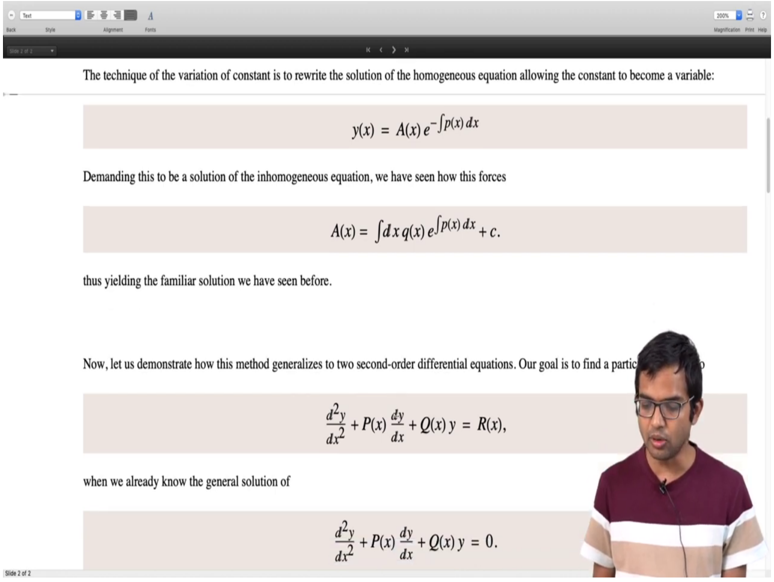

And then you compute this quantity y of x is equal to A of x times e to the minus integral p of x dx right, you this part you just leave it as it is, but instead of just a constant A you make it a function and you implant this as an ansatz into your the differential equation which you want to solve namely the inhomogeneous differential equation right.

And now you have seen how this works out right you can go back and review this older lecture where we do this in great detail. So, I mean you just plug back in here and then you can you know solve for you know if you force this to be a solution of this differential equation.

And then it will give you a of x a of x is given in terms of this integral da dx q of x e to the integral p of x dx plus constant and once you have A of x you just you know multiply this with this exponential minus integral p of x dx and you are done right. So, this is you know you have seen this same solution.

The final solution obtained by other methods too is the same but, the key point to observe here and to build on is you know this method where you started with the homogeneous differential equation and worked out the solution. And then where you had this constant free constant if you make that into a function and it was possible to use that as an ansatz in a meaningful way to work or to get a particular solution for the inhomogeneous differential equation right.

And it turns out that this method generalizes even to higher order differential equations. So, in particular we will look at second order differential equations in this discussion. So, our goal is to start with a differential equation of this kind. So, you have d squared y by dx square plus P of x times dy by dx plus Q of x times y is equal to some arbitrary function R of x which lies on the right hand side right, which we have seen is sometimes called the forcing term.

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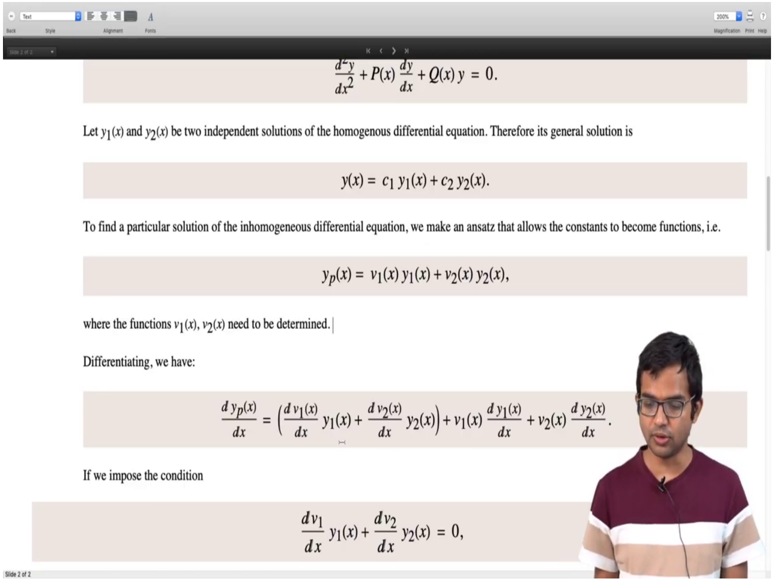

Now, the goal here is if suppose by some means we know the full solution of this differential equation right which itself may not always be an easy problem right. So, because here we are looking at a scenario where both P of x and Q of x may be arbitrary functions right. So, which by itself may turn out to be a hard problem right, we have seen of course, that if both P and Q are constants then we know how to solve it right.

So, that is a solved problem but, even for that scenario if you put if you make R of x to be extremely complicated as of now we do not know of a systematic method to work out a particular solution right. So, which is you know the point of this discussion is if by sum means if we are able to get the full general solution you need the full general solution of this homogeneous differential equation.

It is not enough if you have some particular solution right, if you have the full general solution of this homogeneous differential equation. So, now of course, there are going to be two of these right. So, there are two independent solutions. So, you in general have a general solution which will have two free constants.

So, you will have something like, y of x is equal to c 1 y 1of x plus c 2 y 2 of x where both y 1 and y 2 are you know solutions of this. So, and they are in linear independent solutions which you can couple in this manner and write down the full general solution. So, inspired by our you know first order differential equation and how you know this variation of parameter method works out we elevate these coefficients c 1 and c 2 to the status of functions right.

So, suppose we consider an ansatz we make this y p of a particular solution ansatz as nu 1 of x or v 1 of x times y 1 plus v 2 of x times y 2 of x. Where in place of these constants c 1 and c 2 we have some functions which are to be determined right. So, there is some freedom about these functions, but it does not matter which particular function you find if you can find one particular function we are done right.

So, we have seen this in the past that if you can by some means, but get any particular solution you can just add the complementary function and then you are done and complementary function is of course, available here which is the solution of this homogeneous differential equation.

So, now making this the ansatz if you want to plug this into your original differential equation, you must find the derivative so if you take a derivative of this function you have dv 1 by dx times y 1 plus dv 2 by dx times y 2 plus v 1 d by 1 by dx plus v 2 d by 2 by dx. So, there are all these terms which come in four terms.

Now, in order to simplify the algebra and in fact it is an educated you know simplification right we know you know because, it works it has been tested out by somebody and they have shown that if you make this assumption it can you know it can yield for you a particular solution. So, we make this assumption as well because, we have the you know wisdom from hindsight somebody else has already done it and told us right so it is available in the textbooks right.

So, the standard technique here is. So, since we have this freedom you know v 1 and v 2 are not you know not so constrained that you do not have wiggle room. So, In fact you can choose a v 1 and v 2 to be anything as long as y p is a particular function. So, it turns out that if you make this assumption you impose this condition somewhat arbitrarily, but the justification is that it will work out as we will see right.

So, you have to, you know, use some intuition to come up with a constraint of this kind. So, we make this condition dv 1 by dx times y 1 plus dv 2 by d x y by 2 equal to 0. If you put this condition right so, what it does is it will greatly simplify the second derivative right. So, after all we have to find the second derivative of this yp and then plug back in into your original differential equation and make sure that it all agrees.

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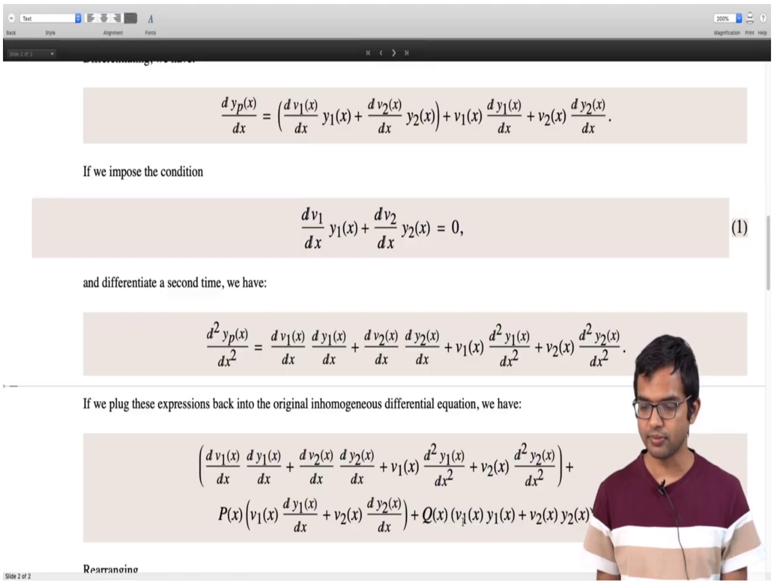

So, in this case if you make this assumption, you know that these two terms will go first two terms and then your d yp by d x is simply given by v 1 times dy 1 by dx plus v 2 times dy 2 by dx. So, if you take another derivative now, you have dv 1 by dx times dy 1 by dx plus dv 2 by dx dy 2 by dx plus then you have these terms involving the second order derivatives right.

So, what this condition has ensured is that there are no second order derivatives in v 1 and v 2 right. So, we want to work with just first order derivatives in these unknown functions. So, that is what this condition does for us. As we will see in a moment this is a simplification which allows us to in fact solve for v 1 and v 2 ok.

So, if you have v 1 of x times d squared y 1 by d x square plus v 2 of x times d squared y 2 of x by d x squared. Now we must use all these ingredients. So, we have this long complicated expression, but not as long as it would have been if you had not made this assumption expression for the second derivative then we have an expression for the first derivative, which is actually simpler than what I am showing here because the first two terms will be absent.

So in fact, it will have only just two terms and then we have an expression for y p of x which is of course, the original ansatz. If we use all of this information and plug back into this original differential equation, then you know in this exact manner if you add all these three terms it must add up to R of x that is our condition that is our requirement for y p. So, that will give us you know differential equations in v 1 and v 2. So, let us go back and plug these into the original inhomogeneous differential equation.

So, now we have you know this whole stuff is the second derivative of y p d squared y p by dx square you know I have put down all these four terms, then plus P of x times the first derivative. The first derivative is just you know these guys this part alone because, the first two terms add up to 0 according to the way our condition has been imposed. So, I have P of x times v 1 of x times dy 1 by dx plus v 2 of x times dy 2 of x by dx right.

So, it looks complicated, but it is really not very difficult. You just have to follow the algebra carefully plus Q of x times the original expression for y p right. So, all I am doing is plugging in my ansatz with the conditions into this differential equation and I have to make sure that it works out. So, I have Q of x times y p, y p is v 1 y 1 plus v 2 y 2 and which is exactly what I have Q times v 1 y 1 plus v 2 y 2 now all of this must equal R of x.

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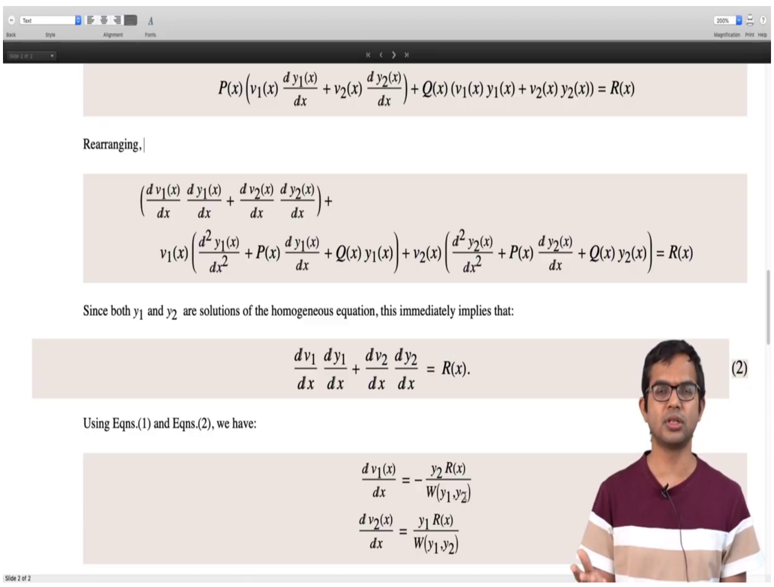

Now, it is convenient to rearrange all this complicated stuff in this manner. So, I put together dy dv 1 by dx times dy 1 by dx plus dv 2 by dx times dy 2 by dx this is the first two terms I just leave it as it is I put one bracket around them then, I collect all terms where there which has a factor of this v 1.

So, I have v 1 of x times d squared y 1 by dx squared plus P of x times dy 1 by dx plus Q of x times y 1 of x each of these terms has a factor of v 1 of x which I pull out and then there are these three terms which have factors of v 2 again. I find that this is v 2 of x times d squared y 2 of x by dx square plus P of x d y 2 of x by d x plus Q of x times y 2 of x and that is it. I have exhausted all terms. I have to just rearrange them in a nice useful way which is equal to R of x.

Now a moment's thought you know if you look at this equation carefully you see that in fact, you know this term and this term the or the entire stuff in the brackets of the second term. And the next stuff this bracket term both of these must vanish this quantity and this quantity must vanish because, after all what are y 1 and y 2, y 1 and y 2 are solutions of the original homogeneous differential equation.

And this is the homogeneous differential equation. So, we started with the assumption that y 1 and y 2 are independent solutions, but basically the solutions of the homogeneous differential equation therefore, in fact we have this great simplification which immediately follows.

So, now we have just two terms here. In fact, the left hand side with all these terms you know canceling away you just get dv 1 by dx times dy 1 by dx plus dv 2 by dx dy 2 by dx is equal to R of x. So, now, I have numbered this as equation 2 right. So, let us go back and look at equation 1. So, equation 1 is a condition we put in by hand. So, I said dv 1 by dx y 1 plus dv 2 by dx y 2 equal to 0.

And now, you see if your v 1 and v 2 are arranged in such a way that y p is a particular solution of your original inhomogeneous differential equation. And this condition one holds we have managed to show that even condition two holds which is another linear equation. So, we have two linear equations in two variables and the two variables being dv 1 by dx and dv 2 by dx because, y 1 and y 2 are known functions R is also a known function.

So, the only thing which is unknown is dv 1 by dx and dv 2 by dx. Of course, it is a 2 by 2 linear problem. We know how to solve it and the answer is simply dv 1 of x by dx is equal to minus y 2 times R of x divided by the wronskian. So, it turns out that this Wronskian appears because of this nice manner in which these derivatives come here. So, you see on the one hand here you have y 1 and y 2.

But, here the coefficients are the derivatives dv dy 1 by dx and dy 2 by dx therefore, you get dv 1 by dx is equal to minus y 2 times R of x divided by W of y 1 y 2. So, the wronskian you know is the determinant of this function.

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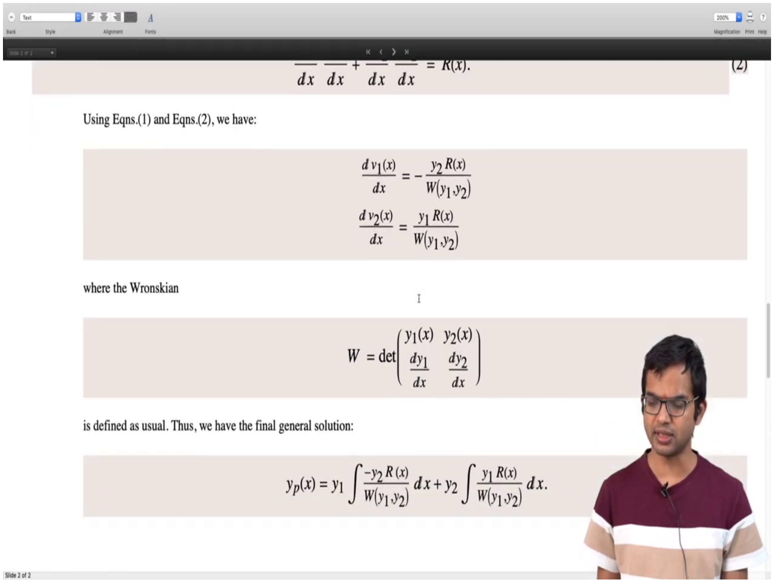

So, you will have y 1 times d by 2 by dx minus y 2 times dy 1 by dx right. So, this is something that you can verify so, dv 1 by dx is equal to you know this expression. And then we also have dv 2 by dx is equal to this expression y 1 times R of x divided by the same wronskian ok.

So, now you know these are linearly independent functions. So, we are guaranteed that the Wronskian is not 0 right. So, it is a useful observation to make it like a side remark, but it is important right. So, we have chosen these y 1 and y 2 to be independent linearly independent solutions therefore, we are guaranteed that their wronskian is non 0 and therefore, these functions are going to be meaningful.

So, formally you can just integrate them and then you have v 1 of x is just the integral of this with respect to dx and likewise v 2 is the integral of this with respect to dx. So, you will of course, get some constant right. So, each of these integrals will give you some free constant, but in fact it does not matter for our purposes.

Because we are only interested in a particular solution and the particular solution we saw, we started with the ansatz y p of x is equal to v 1 of x times y 1 plus v 2 of x times y 2 and we have solved for this. At least formally v 1 is just integral minus y 2R of x divided by W dx and v 2 is integral y 1 R of x divided by W of y 1 comma y 2 d x right.

So, this is the final answer as far as the particular solution is concerned. Once we have the particular solution of course, we just add the complementary function and then we have the full general solution for this problem right. So, formally it looks a little dense what we have done, but in fact, it is just a clever and you know elegant way of working out a general solution for an inhomogeneous differential equation, if you know the general solution for the corresponding homogeneous equation.

And at least formally you can write down one particular solution and then you are done. So in fact, this method is more general than what we have shown and it will it can be carried through to even higher order differential equations, but that is something that we will allow you to play and try to you know figure this thing out on your own or you can cook up your own differential equation and try to solve it using this method. So, but let us look at an example of how this you know plays out in practice.

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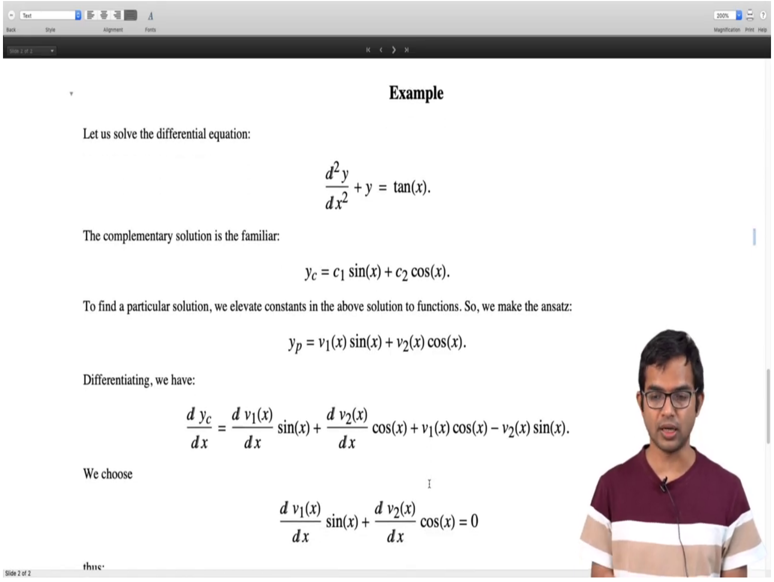

You are drawn from a second order differential equation, which is actually very simple to write down. So, you might think that this should have a very simple solution, but not unless you use the method that we have described here right. So, we know of course, how to solve this is just like a harmonic oscillator problem, but where the external drive is a tan function if it is a sine function or a cosine function right.

We know how to work with this, we also know exponentials, we have our you know algebraic functions so on, right. So if you have x squared x to the power 4 whatever it is we have given prescriptions for problems of this kind. So let us see what happens if you have tan of x. So, the complementary solution is the familiar. So, that is you are solving for d squared y by dx squared plus y is equal to 0.

That is the simplest differential equation, perhaps the first differential equation we have seen whose solution we are aware of from high school days which is just c 1 sine of x plus c 2 cosine of x. So, both the sine and the cosine will appear now, to find a particular solution for this differential equation using the technique above we will elevate these coefficients to the status of functions.

So, in place of c 1 we will put v 1 of x and in place of c 2 we will put v 2 of x and then implant this ansatz y p as a particular solution into your original inhomogeneous differential equation. So, differentiating we have the first derivative is dv 1 by dx times sine of x plus dv 2 by dx times cosine of x plus v 1 of x times cos x minus v 2 of x times sin x right.

So, now, comes the you know choice that we make right exactly like in the prescription the general prescription which we gave I mean, it is useful to work this out explicitly every time rather than try to memorize you know some solutions in terms of the wronskian and all this it is nice for a formal general prescription.

But it is best to work it out explicitly every time we have a you know problem of this kind. So, that is why I am giving you an example and working it out in somewhat elaborate fashion. So, we choose dv 1 by dx times sine of x plus dv 2 by dx times cosine of x to be 0 right.

So, that is the prescription. Which has been just handed down to us and then we will and then we argue that it works and therefore, it is useful to adopt it - we have seen that it works.

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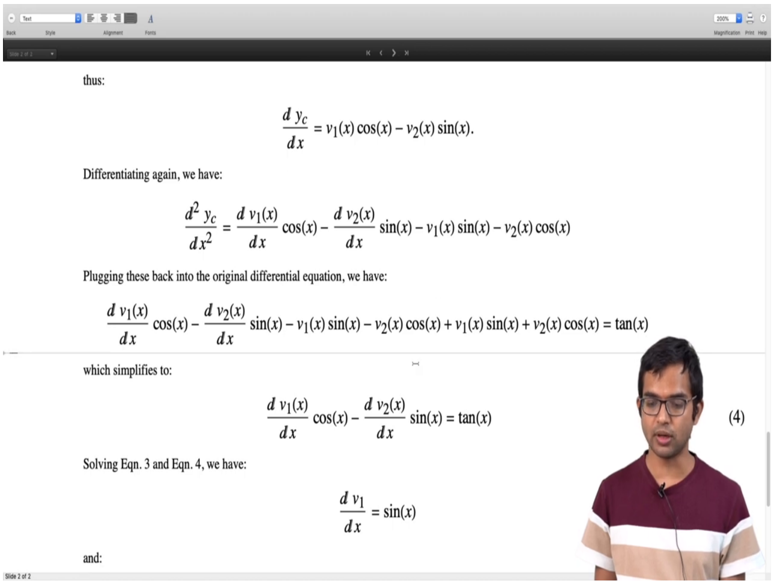

So, dv 1 by dx sine of x plus dv 2 by dx times cosine of x equal to 0 thus, d by c by dx is we have a simplified expression so it is simply v 1 of x times cosine of x minus v 2 of x times sine of x. So, the first two terms you know add up to 0. So, if you differentiate once again.

So, we have dv 1 by dx cosine of x minus dv 2 by x by dx times sin of x minus v 1 of x times sin of x minus v 2 of x times cosine of x. Now we have to plug all of this back into the original differential equation, which is actually a very simple differential equation you have to just add the second derivative and the function itself.

And it must give us tan of x and the function is chosen to be this y p. So, we take this you know this complicated looking stuff, but it is actually not so complicated and in fact, there will be simplification as you will see so, you have all these 1, 2, 3, 4 terms. So, you know first derivative times cosine of x minus dv 2 by dx times sine of x minus v 1 sine of x minus v 2 of x cosine x plus.

So, this is where the simplification comes when you add the function itself: it is v 1 of x times sine of x plus v 2 of x times cosine of x. So, all these last four terms simply cancel and then we must add these all up whatever is remaining, which is just two terms they must add up to give us tan of x.

So, the differential equation that we have well I mean, it is the second differential equation we have d v1 by dx cosine of x minus d v2 by dx sine of x equal to tan x. So, you can think of these equations 3 and 4 as you know coupled equations. So, you have actually two linear two you know first order equations: there are two unknowns v 1 and v 2 and you have to solve for dv 1 of a by of x by dx and dv 2 of x by dx separately. In fact, each of them will be separable as we have seen right.

So, it is not a you know not coupled differential equations in their full complexity, but you know very straight forward.

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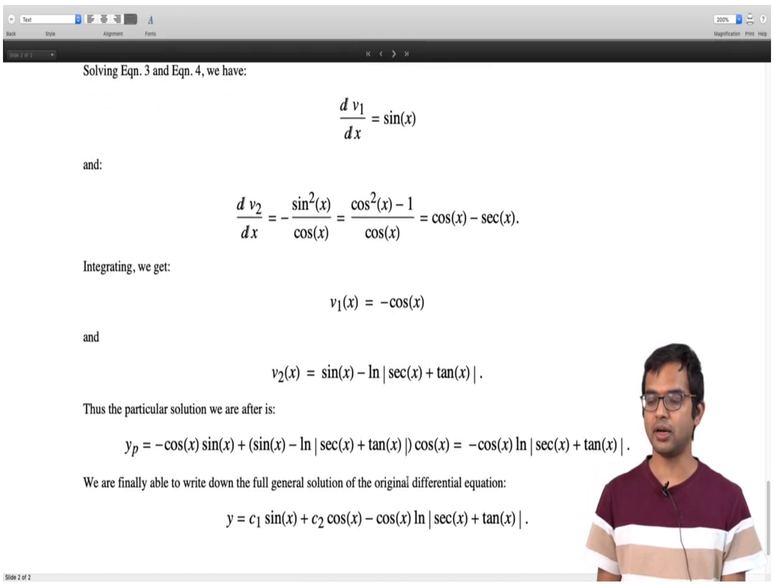

So in fact, you can write down d v1 by dx immediately you can solve for this and you get sin of x right you can if you wish verify that the you know wronskian is playing out and so on, but it is you know it is straightforward to directly solve for each of these unknowns dv 1 by dx is equal to sin of x and dv 2 by dx is minus sin squared of x divided by cos x. You can verify that indeed you know if you plug this back in here.

This times cosine x times cosine x minus you know plus sine squared x by cos x. So, which will just add up to tan x. So, where you just have to make use of the fact that cosine squared plus sine x will give you 1. So, sin x y cos x and is equal to tan x. So, it works out alright. The signs are all correct.

So, d v2 by dx is equal to cos squared x minus 1 by cos x. It is more convenient to write it this way so, cos square x y cos x is cos x minus secant of x. So, now integrating the first equation is straightforward so, v 1 of x is minus cos of x plus an arbitrary constant, but we are after a particular solution.

So, we do not care about the constant we might as well just put it to 0 v 1 of x is equal to minus cos x and v 2 of x. So, the first term will give us a sin of x minus log of secant of x plus tan of x right. So, again there is an arbitrary constant which we just put it to be 0 because, we do not worry about this constant here we are after a particular solution.

And the particular solution also is simplified now you see. So, I have y p is minus. So, let us recall what we were after, the ansatz for the particular solution was c 1 v 1 of x times sine x plus v 2 of x times cosine of x v 1 of x is minus cos x. So, minus cos x times sin x plus this whole stuff for v 2 sin x minus log of secant of x plus tan x the whole thing must be multiplied by cosine x.

So, now you see that there is going to be simplification. So, cosine x minus cosine x minus sine x will cancel with sin x cosine x and then we are just left with this one term, which is minus cosine x log of secant x plus tan x our rather compact simple expression for the particular solution after all the hard work.

So, we can in fact go ahead and write down the full general solution for this original differential equation namely you know this rather simple looking, but as you have seen which involves some careful analysis this differential equation, d squared y by dx square plus y is equal to tan of x the final answer is just the complementary solution c 1 sin x plus c 2 cos x minus cosine of x times log of secant x plus tan x.

Which is your particular solution, which we have just worked out using the method of variation of parameters. So, you should play more with you know other kinds of functions you can put in place of tan x you come up with something more complicated and see, whether you know you may run into difficulties with integrating this but in principle this method holds and then if you can do it with second order differential equations you can also try it out with you know maybe order three differential equations.

If you like or you know do some small improvisations within this thing it is a you know simple elegant and beautiful method and hopefully I managed to convince you on this.

Thank you.