Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 67
An Alternate Approach to the Auxiliary Equation

So we have seen how a homogeneous second order differential equation which is linear and with constant coefficients can be solved when you look at the auxiliary equation, find its roots and then basically you are able to write down the solution for the problem.

Now in this lecture we will basically revisit this procedure, but you know give a slightly different way of thinking about the same thing right you can think of this as a you know the essential results would be a repetition, but it is just an alternate perspective ok.

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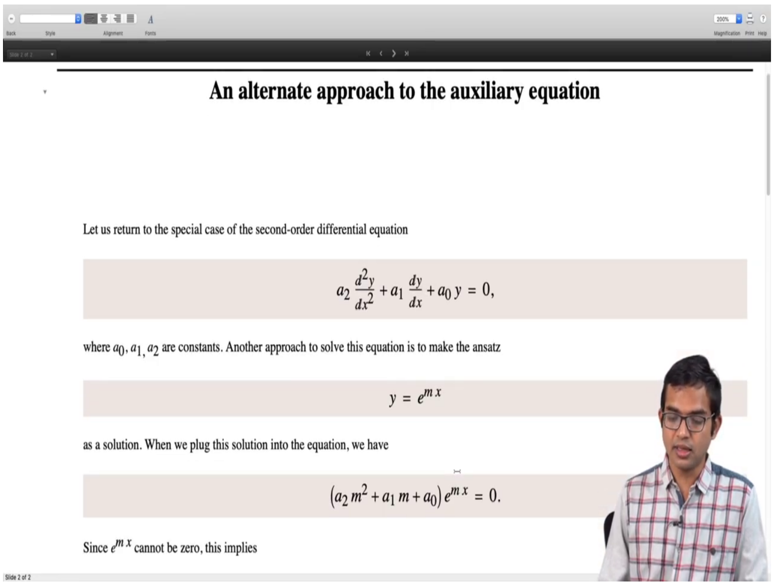

So, the differential equation of interest is a 2 d squared y by dx squared plus a 1 dy by dx plus a naught y is equal to 0 right.

So, we said you know you write it as this differential operator you know think of this these as you know manipulate them as if they are like regular variables and then write it as you know a quadratic expression, and then whose roots we find and you know depending on the roots of the corresponding quadratic equation we were able to write down the solution right.

So, where does this auxiliary equation come from?. So, another way of thinking about this is to start with the differential equation and make an ansatz right. So, you make a guess for the solution for this differential equation and you say suppose y is equal e to the mx is a solution for this problem. Can we find an m such that y equal to e to the mx is a solution for this differential equation right.

So, if you do this then you see that when you take a first derivative of this you get just m times y and if it when you take a second derivative of this function you get an m squared times you know the same y or e to the mx right. So, when you plug; plug this into this differential equation you get an algebraic equation of this kind you get a 2 m squared plus a 1 m plus a naught times e to the mx. So, e to the mx does not matter right so, because this must hold for every value of x.

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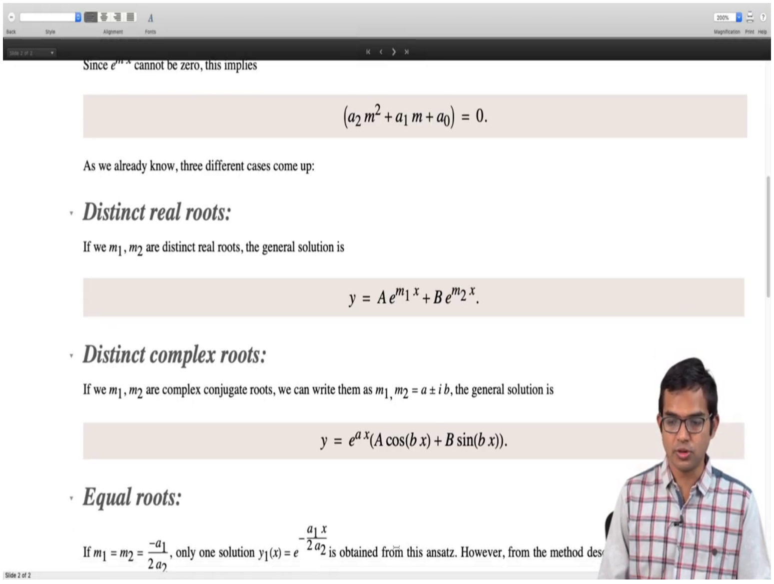

So, basically we are interested in finding m such that a 2 m squared plus a 1 m plus a naught must be equal to 0 right. So, this is nothing, but the auxiliary equation right. So, it is a quadratic equation involving these coefficients and so, that is how the auxiliary equation come comes up right and so now, we can look at the 3 different cases we know that a quadratic equation the solution of a quadratic equation you know there are 3 cases which one can consider.

One is if you have distinct real roots if you have distinct real roots m 1 and m 2 then no problems at all. So, we have managed to find 2 solutions e to the m 1 x and e to the m 2 x since m 1 and m 2 are distinct and real you can if you wish check that you know y 1 and y 2 are linearly independent solutions by finding out the Wronskian and checking that its nonzero, but you know these are you know distinct and linearly independent solutions.

So in fact, the general solution for this problem is simply y is equal to a times e to the m 1 x plus b times e to the m 2 x. So, it's a linear second order differential equation. So, it cannot have more than 2 linearly independent solutions. It has exactly 2 of them and so, these 2 coefficients which are arbitrary coefficients A and B which will be fixed depending upon the initial conditions and boundary conditions right ok. So, the second possibility is if you have distinct, but complex roots right. So, this is somewhat similar.

So, in place of m 1 and m 2 you write it as a plus or minus ib and then you can basically it's the same solution, but then it convenient to pull out this e to the ax because which is common to both these solutions and so, you can also if you wish write it as e to the ax times A cosine b of x plus B times sin of bx right. So, this is one of sort of many different ways of writing down the same general solution right this is also straightforward.

And then there is a third case which is somewhat subtle and that is when you have equal roots if you have equal roots then well they are going to be real for sure right. So, and they are going to be minus a 1 divided by 2 a 2 right. Well, I mean well we have assumed that a 1 and a 2 are real. So in fact, you could keep it you know completely general a naught a 1 and a 2 are all constant. So, they can be real or imaginary if you wish, so that is fine.

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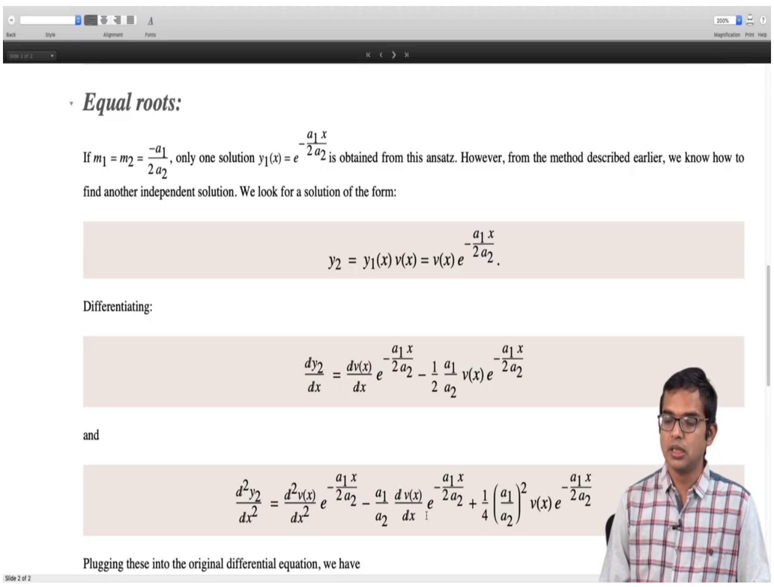

So, if m 1 and m 2 are equal right. So, then you have only one solution and that one solution is just given by minus a 1 divided by 2 a 2 right. So, how do you find the other linearly independent solution right? So, we make use of the technique we discussed in the previous lecture right.

So, there it was of course, a much more complicated differential equation right we had you know we allowed a 2 to be a 2 of x we allowed a 1 to be a 1 of x and a naught to be a naught of x right. So, if we could find one solution for such a differential equation we had said how there is this method by which you can find the linearly independent solution right.

So, but in this case we have found one solution which is y 1 of x is equal to e to the minus a 1 x divided by 2 a 2 right. So, we will use the same technique and then look for a second solution of this form. y 2 which is y 1 of x times v of x. So, you have v of x times e to the minus a 1 by 2 a 2 you know times x here right. If you plug this back into your differential equation.

Now so, you have first we have to find dy 2 by dx which is going to be dv you have to carefully collect all these terms there are couple of terms you know all these factors and signs you have to be careful dv by dx times e to the minus a 1 x divided by 2 a 2 minus the half times a 1 by a 2 times v of x times e to the minus a 1 x divided by 2 a 2.

So, if you take the second derivative you get one more term right. So, you get d squared dy by dx squared e to the minus a 1 x divided by 2 a 2 minus a 1 by a 2 dv by dx e to the minus a 1 x by 2 a 2 right. So, this is a. So, this comes from taking the derivative of this this part of this exponential so that is going to give you one of these and the other one is going to come from you know taking a derivative of you know this part dv by dx if you add these 2 you the half and half will add up to 1.

So, minus 1 a 1 by a 2 dv by dx e to the minus a 1 x by 2 a 2 then you get one more term which is obtained when you take the derivative of this exponential here. So, v of x remains as it is the exponential derivative that the factor remains as it is, but you get a one-fourth a 1 by a 2 the whole squared.

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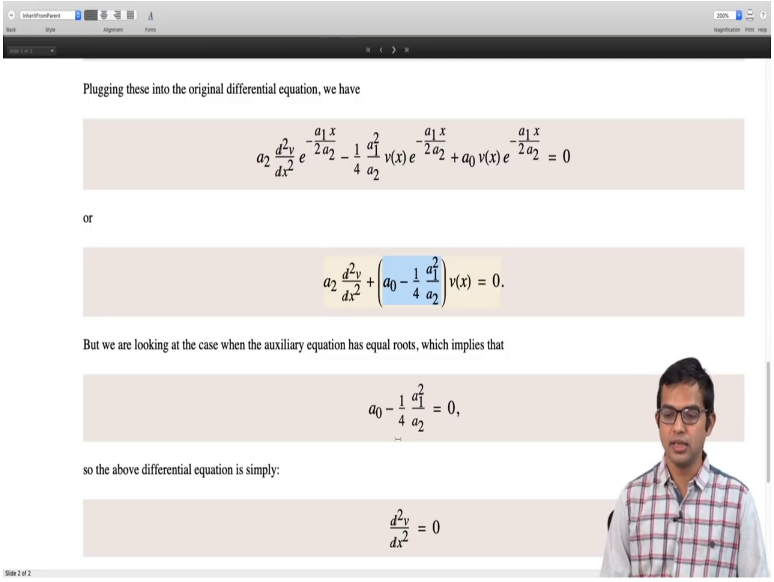

So, you should plug all of these terms you know these 3 objects back into the original differential equation. Then you know there is this simplification which comes about you know because of these in between terms and then you get only 3 terms a 2 d square. So, that second derivative in v of course, exponentials remain as they are minus half a 1 squared by a 2 times v times its exponential plus a naught v times its exponential is equal to 0.

It's a common exponential and this must hold for all values of x so you can just remove that part and when you collect the second and the third terms which are really you know so you just get a naught minus one-fourth times a 1 square divided by a 2 times v of x right the second term.

So, the first term is of course, just a 2 times d squared dy by dx squared the sum of these 2 terms is 0. Now some thought reveals that in fact, because we are looking at equal roots right so there is a; there is a constraint imposed by the fact that we have equal roots and the constraint is that the discriminant of this quadratic equation must be 0. And what is the discriminant of this quadratic equation it is just minus a 1 so its b squared minus ac. So, that is a squared minus b squared minus 4 ac.

So, it is a 1 squared minus 4 a 2 a naught must be equal to 0 right. Which immediately implies that this coefficient or which tags along with v vanishes therefore, we are left with just one term so, the equation is just simply d squared v by dx squared is equal to 0 right.

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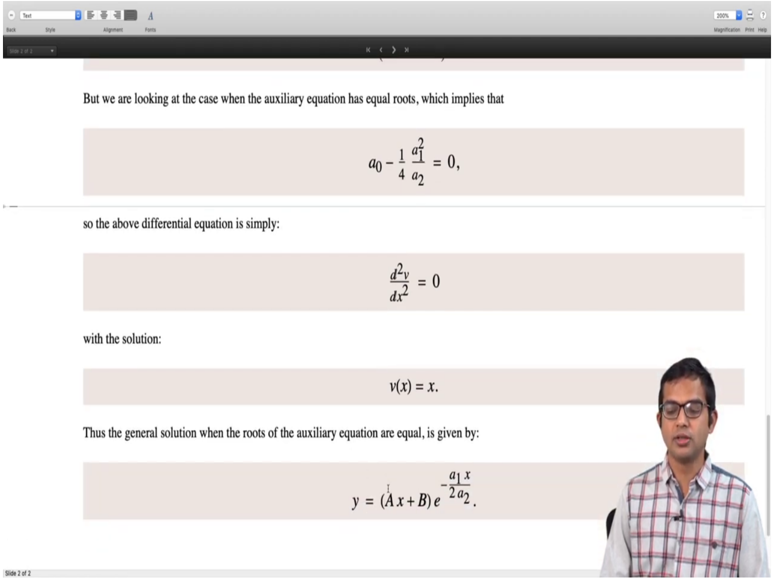

And the solution for this is completely trivial it is just v of x is equal to x right. Which is already the solution which we have used right. It's not a surprise because we have already seen that one solution is e to the minus a 1 x by 2 a 2, and the other solution is x times e to the minus a 1 x by 2 a 2 right.

We have already seen this, we have explicitly verified this, but this was an opportunity for us to use a technique which we have learned which is much more general than we are applying for in this special case. But it is useful to see the same ideas but from multiple directions so that we our understanding is solidified.

So, what have we learnt from this so when you have equal roots we have managed to show that the general solution is given by y is equal to ax plus b times e to the minus a 1 x divided by 2 a 2 right. So, we have recovered using this somewhat you know an alternate approach the same results for this homogeneous second order linear differential equation with constant coefficients.

And we have made a distinction between you know 3 cases when you have these distinct real roots distinct complex roots and when you have equal roots ok that is all for this lecture.

Thank you.