Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 65
Ordinary Differential Equations

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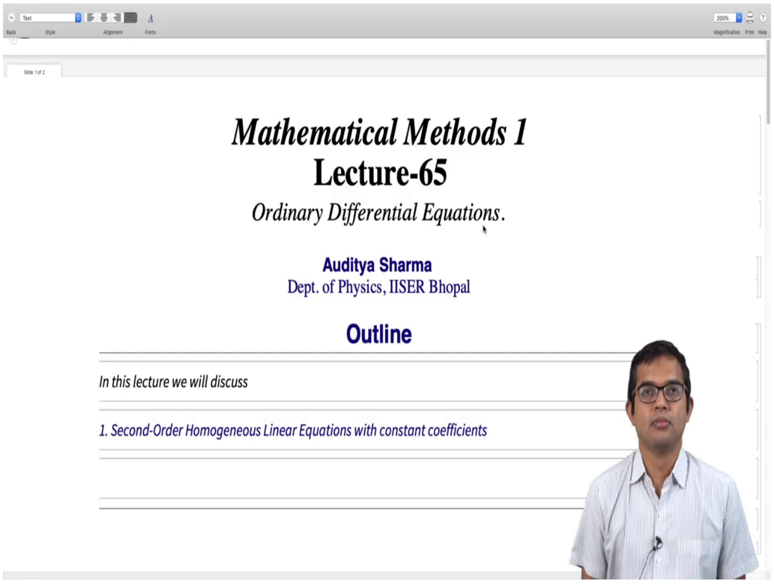

In this lecture, we begin our study of differential second order differential equations, which are genuinely second order in nature and as always, right. So, it's useful to start with the homogeneous version of the differential equation, right. So, later on we will look at inhomogeneous differential equations. So, at the moment we are concentrating on second order linear equations with constant coefficients, right. So, I will give you the explicit form in a moment.

But in this lecture, we are going to discuss second order homogeneous linear equations with constant coefficients; we will lay down the method for solving such differential equations.

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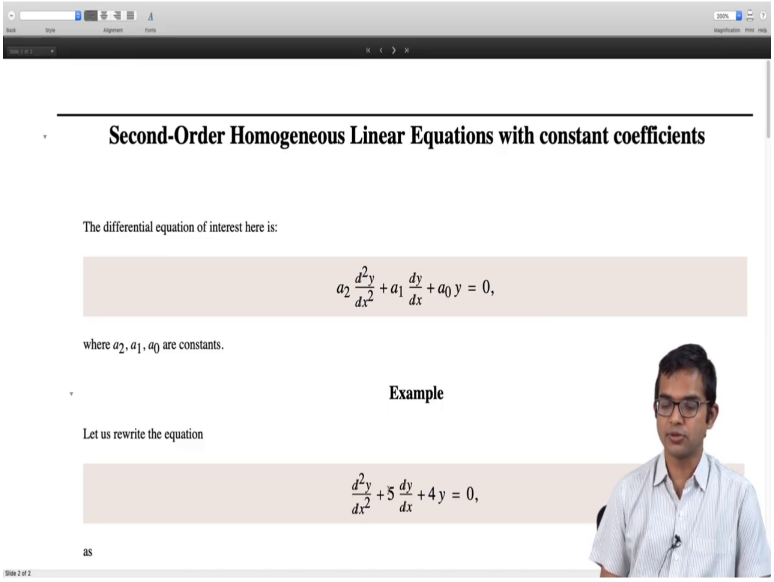

So the differential equation of interest is the following. So, you have a 2 times d squared y by d x squared plus a 1 and d y by d x plus a naught y is equal to 0. So, a 2 a 1 and a naught are all constants and the right hand side is 0, which is what makes it a homogeneous differential equation.

Later on we will see how we can use the solutions from the homogeneous differential equation to work out the general solution of the inhomogeneous differential equation, right. So, we have seen already with first order differential equations, how you know the; even when you are given a an inhomogeneous differential equation, when that is of interest, it is useful to find the solution of the homogeneous in a counterpart of the same equation and then use the solution from there to find the general solution, right.

So, we will see explicitly later on how this works out; but for this lecture, we will concentrate on this homogeneous differential equation second order with constant coefficient. So, let us look at a specific example; suppose you have d squared y by d x square plus 5 d y by d x plus 4 y equal to 0.

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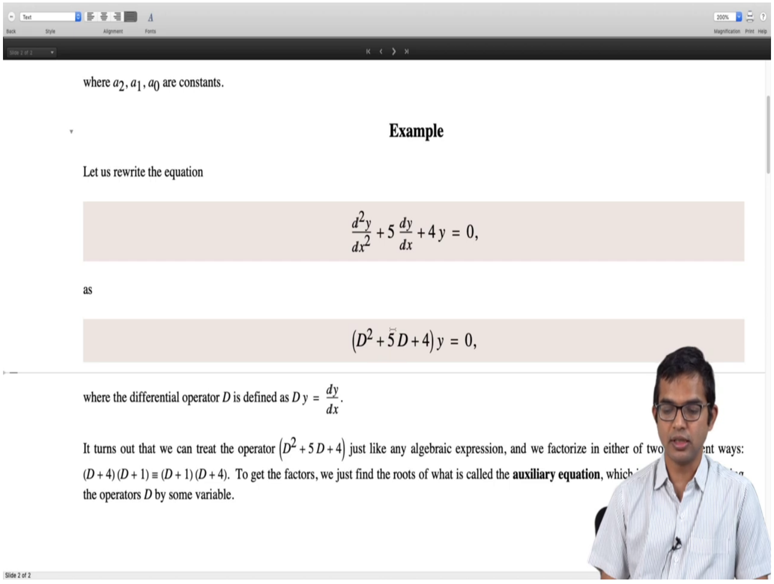

Now, it is convenient to think of this as you know in this operator language. So, you think of this as D square plus 5 D plus 4 acting on y equal to 0, right. So, you have some operator which is acting on y and it gives you 0. And what is this operator? It is the differential operator, right. So, its a linear operator, so it acts on y and it gives you dy by dx. And if you act with another D, that is going to give you the next derivative.

So, that is written as D square right. So, we will see the usefulness of this as we continue this discussion. So, the idea is that, you can actually treat this operator expression as like an algebraic expression and in fact, you can go ahead and factorize this, right. So, there are two convenient ways of factorizing this. So, I mean if you thought of this you know quadratic form D squared plus 5 D plus 4 as this you know, you can equate that itself to 0 and think of this as a quadratic equation, you will get two roots.

And so, that it can be used. So, in this case the roots are 1 minus 1 and minus 4, right. So, therefore, you can factorize, right. So, this quadratic form can be written as D plus 4 times D plus 1, if they were you know usual variables and it turns out that, you know this operator equation itself can be written in this form D plus 4 times D plus 1, right.

And so, the corresponding quadratic equation is called the auxiliary equation and the roots of the auxiliary equation are of great importance in finding the solution of this, which we will discuss ahead. But before we do that, let us quickly familiarize ourselves with the algebra of these kinds of operators.

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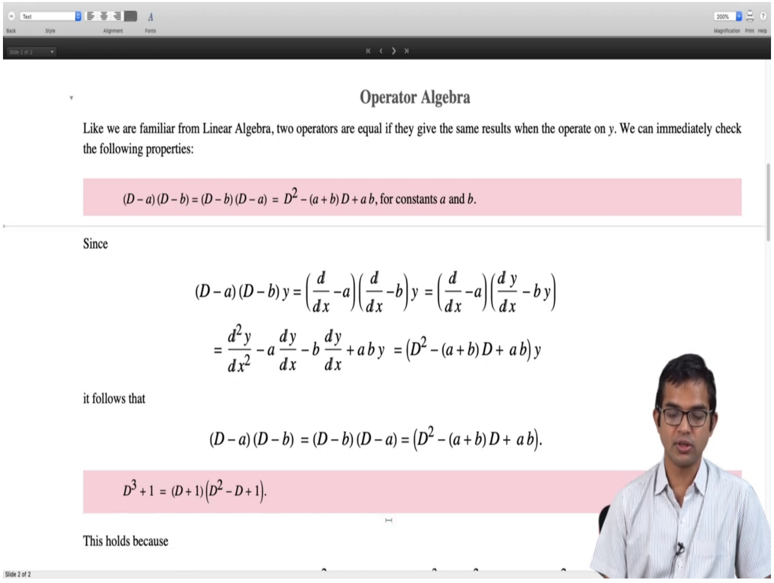

Let us write down a few properties and see how they work out, right. So, we are familiar with linear operators from linear algebra. So, this D is also a linear operator, right. So, all these properties corresponding to linear operators will follow through. Let us check a few properties which are useful for us. So, one is like what we just already said, you know D plus 4 times D plus 1.

It does not matter how you factorize it, D plus 1 times D plus 4 and, D plus 4 times D plus 1 really are the same, right. So, two operators are said to be equal if, you know what they do is the same, right. So, if two operators acting on you know the operators act upon a vector, right. So, here in this case y it, D acts on y and it gives you what is the outcome of this operation.

If it is the same right; if you are given two different operators and whenever it acts on any you know, why they both the result must be the same, when that happens, they are equal. So, the statement here is that, this property that we can show is D minus a times D minus b is the same as D minus b times D minus a is the same as D squared minus a plus b times D plus a b for constants a and b, right.

So, another way of saying this is this, you know these two commute, this operator D minus a, and D minus b right, so that is another way of saying the same thing. So, let us look at how this comes about; you know the way to see this is to operate with this D minus a times D minus b on y. So, that is the same as d by dx minus a times d by dx minus b acting on y. So, this part you know must be made to act on y first. So, then you get d by dx minus a times dy by dx minus by.

So, it is a linear operator. So, there is this property of distributivity. And then once again you use the property of distributivity; so you get, you know you start by operating with d by dx. So, you get d squared y by dx square minus a dy by dx minus b dy by dx plus a b y, you can convince yourself, right. So, each of the terms on the left hand side will act on each of the terms on; if the stuff that is being acted upon, so you get all these four terms.

And then you can collect these terms; so second and the third term are really you know just if they can be summed, so you get a plus b times dy by dx with the overall minus sign. So, you can collect all this and write it in this compact form as D squared minus a plus b the whole times D plus a b, the whole of this acting on y, right. So, you can quickly see that, it does not matter if you had done D minus a times D minus b or D minus b times D minus a, you would get the same.

And both of them would be equal to this D squared minus a plus b times D plus a b, right. So, that is what this property is.

And so, the key point which I should emphasize once again is that, this works out because both a, and b are constant. If they had been operators, then this may not you know such a simple commutation may not hold right, like we will see. So, therefore, we have managed to show that you know this property holds. Let us look at another property, which can be you know one can write a more generalized version of this.

But let us say D cube plus 1. So, instead of D cube plus 1, one could have also thought of this as D cube plus a cube and you are free to work that out that more general version. So, let us look at how D cube plus 1 is equal to D plus 1 into D squared minus D plus 1.

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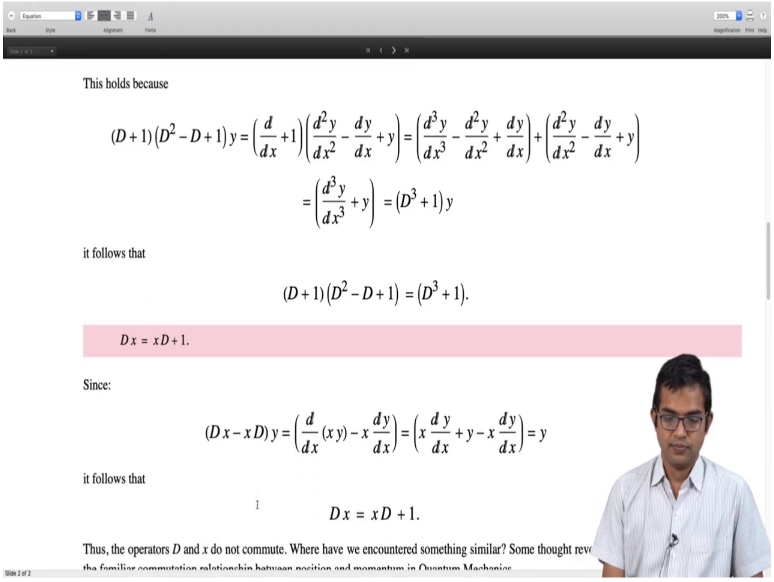

Once again the technique is very similar to the previous property, D plus 1 acting on D squared minus D plus 1 y is equal to d by dx plus 1 times this whole thing; then you expand, then you get all these cancellations, you know the second, third, fourth and fifth terms will cancel.

And then you will be left with just d cube y by dx square d cube y by dx cube. So, this should be x cube. So, you get d cube y by dx cube and here right here d cube y by dx cube is equal. So, this is the whole thing is D cube plus 1 acting on y therefore, it follows that. Since it is true for any y, D plus 1 the operators themselves are equal, right.

So, D plus 1 times D squared minus D plus 1 is equal to D cube plus 1. So, another interesting and useful property is this Dx. So, when the operator D acts on x, you know if you change the order of this; x itself you can think of as an operator, right. So, it is just multiplication xD; then you know they are different, their xD and Dx are different, so in fact you can show that Dx is equal to xD plus 1. So, let us see how this works out.

So, let us operate it with Dx minus xD on y. So, Dx minus xD on y, you know you bring in y. So, D must act on x times y, so that is what is meant by Dx. And so, you must first multiply by x and then D will act on the result of this multiplication. So, D will act on x times y minus x times dy so, but the derivative of x times y is the same as x times dy by dx plus y minus x times dy by dx, overall outcome is just y. So, we see that Dx minus xD is actually just identity, it is just one, right.

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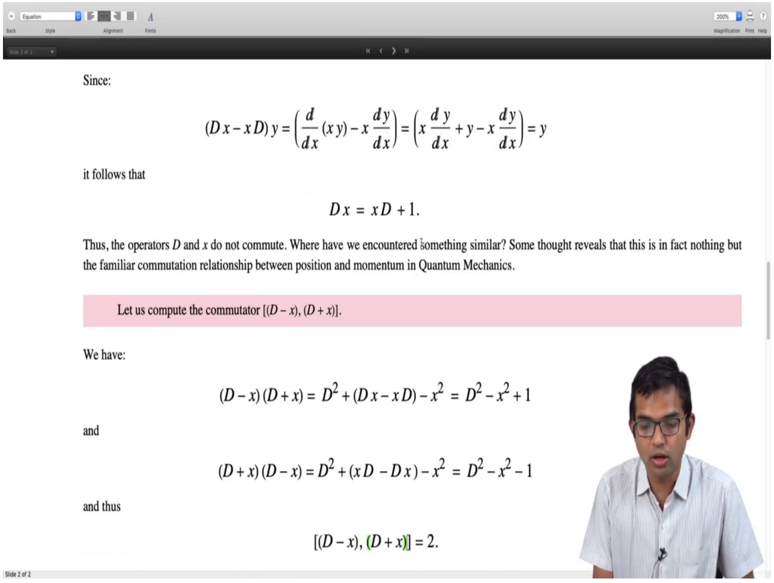

So, thus what we have seen is that, these operators D and x do not commute and this is not the first time you are seeing something like this; in fact you have worked with the same kind of operators. You must have seen in quantum mechanics that, the operator corresponding to position and the operator corresponding to momentum they do not commute.

But what is the operator commuting to position, it is x and the operator corresponding to momentum is like the derivative, which is what we have; with there is a constant y times h cross minus I times h cross it does not matter. But the key point is that, you know there is this non commutativity between x and D. So, which here in this case turns out to be Dx minus xD is equal to 1, right.

So, there you get x comma p in the commutator is I h cross, right. So, that comes because of the factors involved, right. So, this is not unfamiliar at all. So, let us use this to see one more commutator; you can work out more complicated commutators and perhaps that will be part of homework, but just to get a hang of this, you know the algebra around this and how one should exercise care.

Let us look at one more example D minus x times D plus x and D plus x minus D minus x they do not commute, right. It is quite different from D minus a times D minus b, right. So, I said D minus a, and D minus b you know they are the same; but if you take, so I in fact what I mean to say is. So, let us look at this commutator D minus x and D plus x.

So, this commutator. So, it is D minus x times D plus x minus D plus x times D minus x. So, if I work this out D minus x times D plus x is D squared plus Dx minus xD minus x square. But Dx minus xD we have just shown is equal to 1; so I have D squared minus x squared plus 1. So, I mean in regular algebra I would have thought that, you know something like a minus x into a plus x is just a square minus x square; but here you see you get this extra 1 right, which makes all the difference.

And on the other hand you should convince yourself that, if I do D plus x times D minus x; I get D squared plus xD minus Dx. So, there is a minus sign here, minus x squared remains as it is. So, you get D squared minus x squared minus 1. So, if I take, if I subtract these two quantities; so I will get a 2. So, in fact, if I do this commutator D minus x comma D plus x, so I get equal to 2, right.

So, they do not come here, right. So, that is the main message from this discussion, ok. So, after this slide D 2 on operators and how they act on, you know how some of these properties of this operator algebra are concerned; let us get back to our original discussion.

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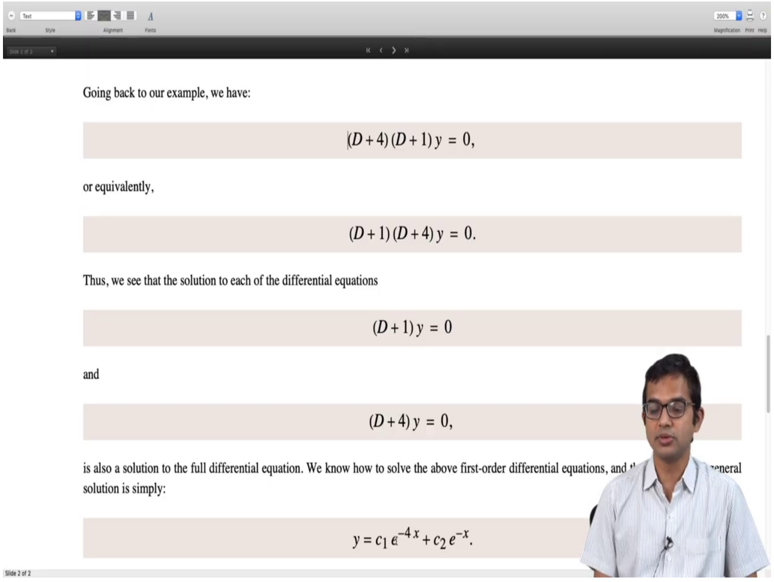

So, we were looking at the second order differential equation D plus 4 times D plus 1 acting on y equal to 0. Now, we have just argued that it does not matter whether I think of this as D plus 4 times D plus 1 acting on y equal to 0 or D plus 1 times D plus 4 acting on y equal to 0. So, we see that the solution to each of these differential equations; if I somehow managed to solve just D plus 1 acting on y equal to 0 or if I solve D plus 4 acting on y equal to 0 right, each of these solutions is also a solution for the full differential equation, right.

So, that is because you know D plus 4 acting on 0 will be 0 and likewise D plus 1 acting on 0 will be equal to 0. So, but each of these is just a first order differential equation, linear first order differential equation homogeneous and we know how to solve it. And so, in fact we can go ahead and write down the answer right; I will allow you to check this, there are these two solutions and they are linearly independent solutions, which you can check if you wish with the help of the Wronskian of these functions.

But for sure there are, these are two linearly independent solutions of a full differential equation and an arbitrarily linear combination of these two linearly independent solutions will be the full general solution of this problem and that is it. So, you see how we manage to, for this case, just solve the first order differential equation and then write down the solution of the full second order differential equation.

So, here it turns out to be c 1 times e to the minus 4 x plus c 2 times e to the minus x, right. So, all we have done is actually just to use the auxiliary equation, find its roots and using that, we are able to write down the solution of the full second order differential equation, right.

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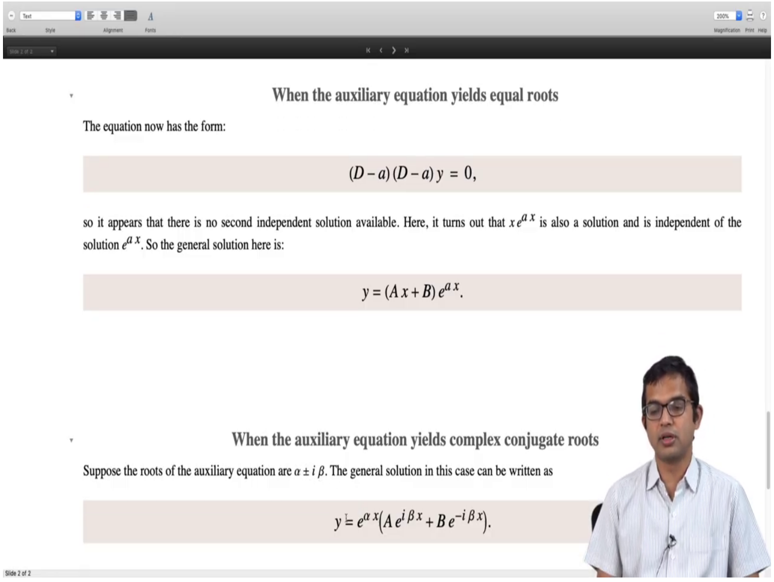

So, the auxiliary equation itself, you know there are a few cases one should consider, right. Suppose the auxiliary equation yields equal roots, suppose you had a scenario where you have D minus a times D minus a times y equal to 0; then I mean it seems like we have only one solution, which is e to the, a x right, only one linearly independent solution.

So, how do you write the general solution for a second order differential equation? So, it turns out the x times e to the, a x is also a solution for this right, as you can use you can check, right. So, I am giving you a prescription here, you can check that this works out. So, in fact the general solution for you know this scenario is given to be, is given by this expression A x plus B times e to the a x, right.

So, you have x times e to the, a x is one solution and just e to the a x is another linearly independent solution, right. The linear independence can also be checked with the help of around skin if you wish. So, then there is a third scenario which also one should be aware of, which appears frequently and that is when the auxiliary equation yields complex conjugate rule, right.

If one of the roots is complex, the other also must be complex right; we know this from you know quadratic equations and indeed the two roots are complex conjugate roots. So, you can have a scenario where the roots are alpha plus or minus i beta. And in this case, the general solution will turn out to be e to the alpha x times A times e to the, i beta x plus B times e to the minus i beta x, right. So, once again you have these two free constants.

So, alpha and beta are already given to you, you have got this from the differential equation in question. So, you start with the differential equation, find the auxiliary equation, find its roots and those roots turn out to be alpha plus or minus i beta. So, there are these you know two numbers and alpha and beta which are known to you and therefore, you just simply write down the solution; y is equal to e to the alpha x times 8 A times e to the i beta x plus B times e to the i minus beta x, so A and B being the free constants corresponding to the second order differential equation.

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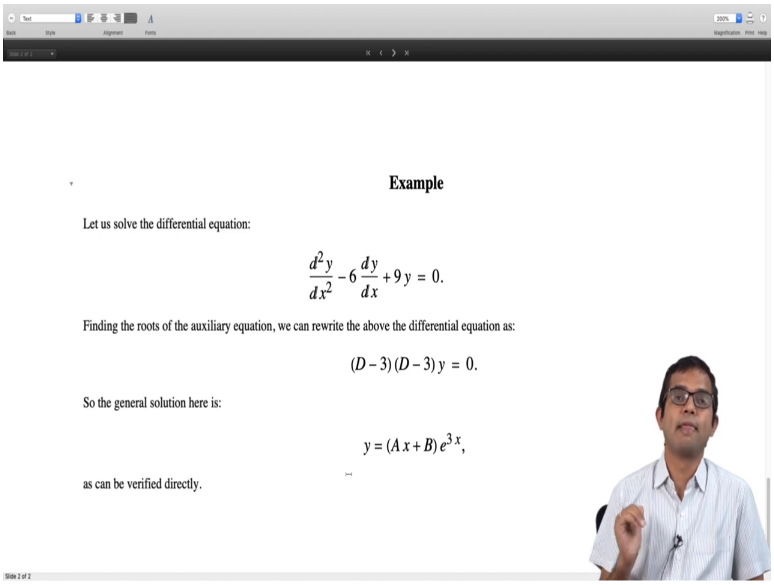

Let us just quickly look at one example. So, if I have this differential equation d squared y by dx squared minus 6 dy by dx plus 9 equal to 0, the roots of this auxiliary equation turn out to be just 3; they are repeated roots, so you get D minus 3 times D minus 3 times y equal to 0. So, the general solution according to the above prescription is A x plus b times e to the 3x as can be verified directly; you just take the final solution and plug it in into your original differential equation.

And you can check that indeed this is the correct solution. And given the fact that it has two free constants, that is all that is as general a solution as you can get; because the second order differential equation will have precisely two free constants, considering that it is linear, right ok.

So, we have in this lecture managed to check out the prescription for solving a homogeneous second order differential equation with constant coefficients, that is all for this lecture.

Thank you.