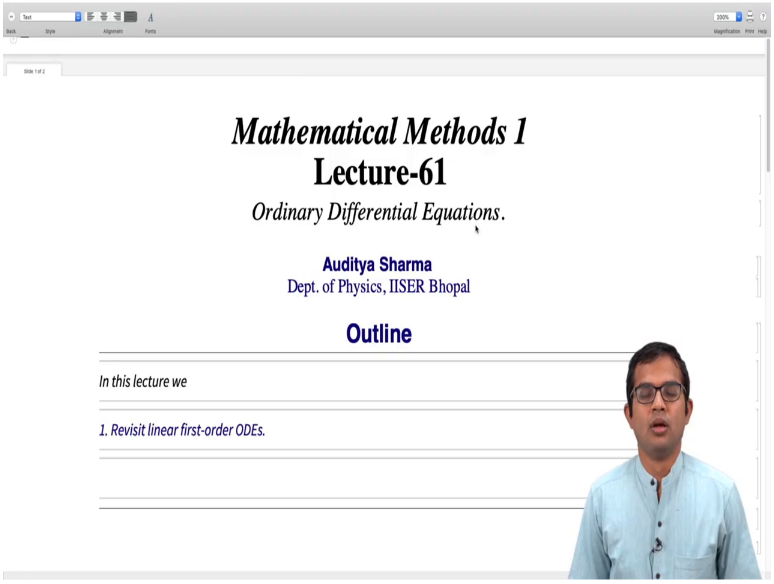
Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Ordinary Differential Equations
Lecture - 61
Revisit linear first-order ODEs



Ok so, we have been discussing ordinary differential equations. We have seen the prescription for solving a linear first order differential equation.

We also discussed when an equation is exact and the concept of an integrating factor, right. So, sometimes an equation is not exact, but it is possible to find a factor you know multiplying this equation throughout with this factor renders it exact and once it is exact, it is easy to solve it, right.

So, let us Revisit linear first order ODEs you know and use our knowledge of integrating factors that is the subject matter of this lecture ok.

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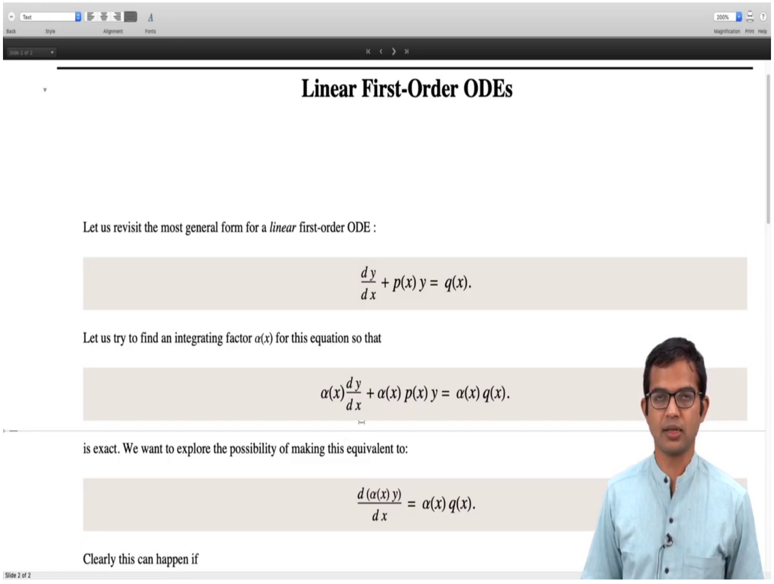

So, the most general first order linear ODE is of this form dy by dx plus p of x times y is equal to q of x, right. So, suppose we want to find an integrating factor for this differential equation. So, what it means is we want to be able to multiply by some function alpha of x throughout and so, the left hand side must become an exact differential, right.

So, we will also explore the specific possibility of the left hand side you know being converted to a form where it can be written as d by dx of alpha times y right I mean we it is a guess that we are doing. So, dy by dx is there. And, then is it possible to multiply by alpha such that basically the left hand side is the same as d by dx of alpha times y, right.

And, if we ask for this right it turns out I mean you have to just expand this and compare the left hand sides of these two equations.

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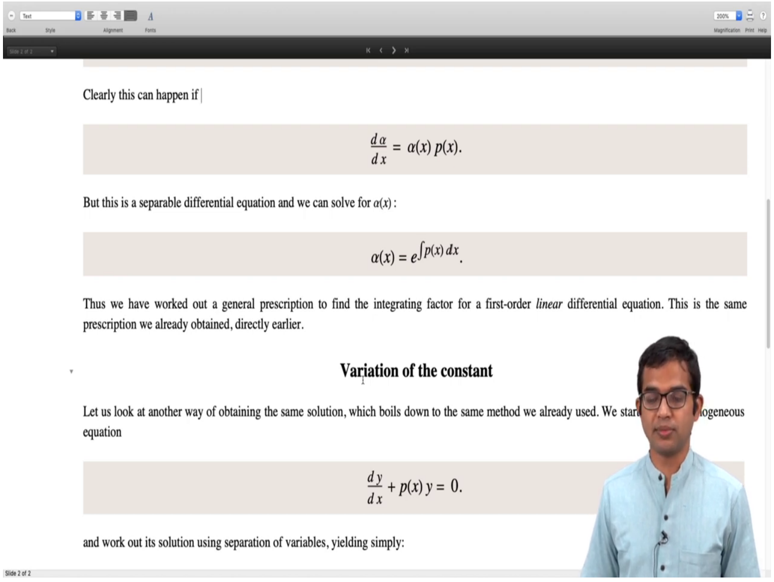

And, then we immediately see that this would be possible if we choose our alpha such that d alpha by dx is equal to alpha times p, right. So, we see that in this equation the first term is dy by dx. It is as if you are treating alpha as a constant, then the second part is where the derivative with respect to alpha comes in and then y is treated as if it is a constant, right.

So, that part if this factor alpha times p is exactly equal to d alpha by dx, then indeed this will hold, right. So, we must solve for an alpha such that you know this condition holds, but a moment's thought reveals that in fact, this is an easy differential equation. We know how to solve this - it is a separable differential equation. So, we just bring alpha to the left side and then you have dx it goes to the right side. So, you integrate, right; at least in principle, we know how to solve this.

So, you have log of alpha is equal to the integral of p dx and then alpha is equal to e to the integral of p of x dx, right. So, this is nothing, but the integrating. So, once we have the integrating factor, we can go ahead and write it: alpha x times y is equal integral of alpha times q alright, dx.

And, then you can write down what y itself is, right. So, if you compare notes with the earlier lecture where we worked out the solution for the 1D for the first order linear differential equation you will see that in fact, you get exactly the same answer as before from this method. But, this is just it is illuminating to think about this in terms of an integrating factor and finding a solution for this integrating factor on this particular form.

So, there is another way of thinking about this and that is the so-called variation of the constant right, which effectively boils down to the same method that we used you know to first find the solution for the problem right. So, that is again let us start with the homogeneous equation.

So, we have dy by dx plus p of x times y equal to 0, right. So, we know how to solve for this right this is a separable equation.

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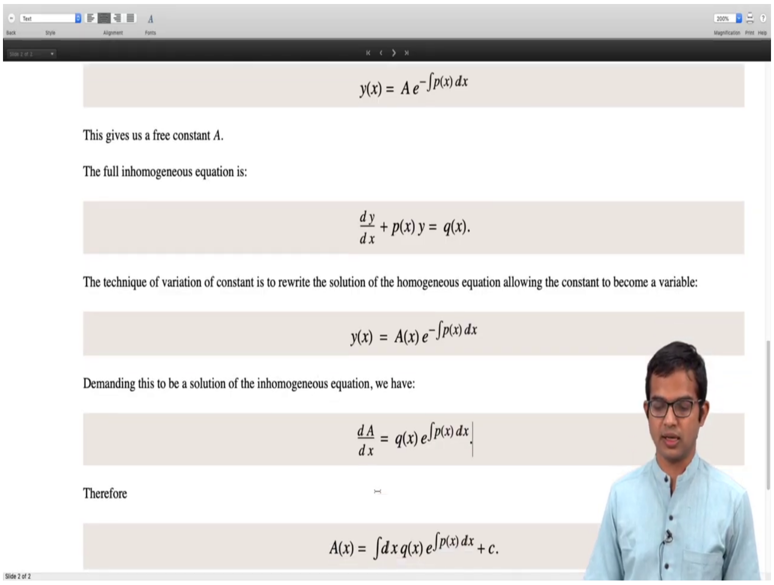

We solved for y and we simply have y of x is equal to A times e to the minus integral p of x dx. So, this gives us a free constant A.

Now, using this we want to be able to find the full solution for the full inhomogeneous equation, right. So, the technique of this variation of constant says that you elevate this constant to a function, right. So, I mean it is essentially what we did earlier, but I am putting it in a somewhat different language and then we will just recover that. It is just you can think of it as a revisiting exercise, alright.

So, the technique of variation of constant is to just write y of x as A of x times e to the minus integral p of x dx and now, this is taken to be an ansatz right. So, an ansatz is a guess that you make for a solution of your you know this differential equation, it is an educated guess. So, given that this is the solution for the homogeneous equation.

So, we make the guess that you know y of x is equal to A of x times e to the minus integral p of x dx will be a solution to the inhomogeneous equation, right. So, what should the function A of x be such that it is possible to find such a function? And, then if you demand this to be a solution of the inhomogeneous equation we find that dA by dx must be equal to q of x.

So, this is now simply a matter of writing dy by dx plus p of x times y is equal to q of x and then plugging it in, right. So, if you just plug this in here you will see that you know e to the minus integral p of x dx you know comes out.

So, the first part will be dA by dx into this and then you know this is going to be this cancellation. And so, you are left with you know dA by dx is equal to q of x times e to the integral p of x dx right. So, this is you know it actually appears on the left hand side with the minus sign and then you bring it to the right hand side.

And, there are these two other terms we just get cancelled out. When you take a derivative with respect to x you know if you treat A of x as a constant that has no derivative then you get minus p of x times A of x e to the minus integral p p dx which is the same as minus p times y which cancels with this.

So, basically you are left with just this equation dA by dx is equal to q times e to the integral p of x dx, right. So, this is an equation we can solve for A right. So, A is now it is just you have to integrate. So, it is just the integral of integral dx q of x times e to the integral p of x dx plus c right.

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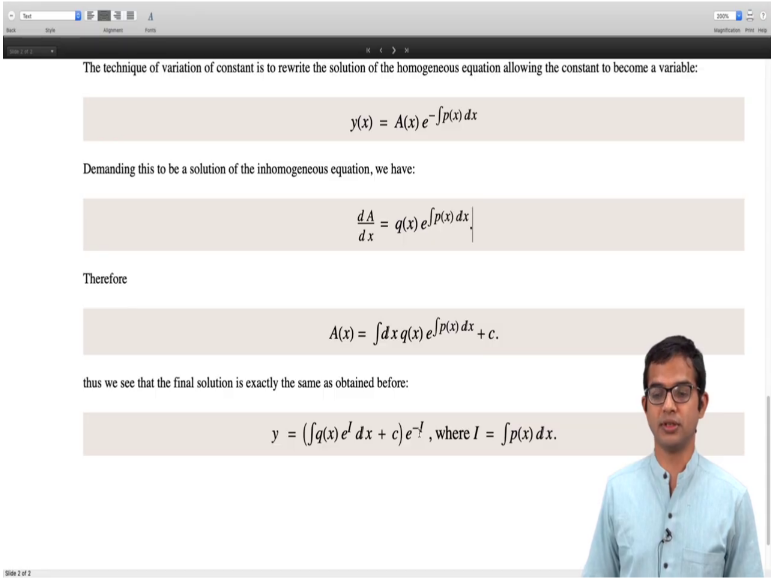

So, that is we see that this is exactly the same solution that we obtained earlier, right. So, y is given to be the integral q of x e to the I dx plus c. The whole you know this entire thing times e to the minus I, where I is integral p of x dx right. So, this is in fact, the same as the first method that we adopted. It is just useful to look at the same problem from many different angles and see that they all come together in a nice way.

So, our understanding of the theory behind this is solidified, right. So, it is in this you know invoking this philosophy we have taken another look at the first order linear differential equation in this lecture, ok.

Thank you.