Mathematical Methods 1
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Ordinary Differential Equations
Lecture - 63
Ordinary Differential Equations

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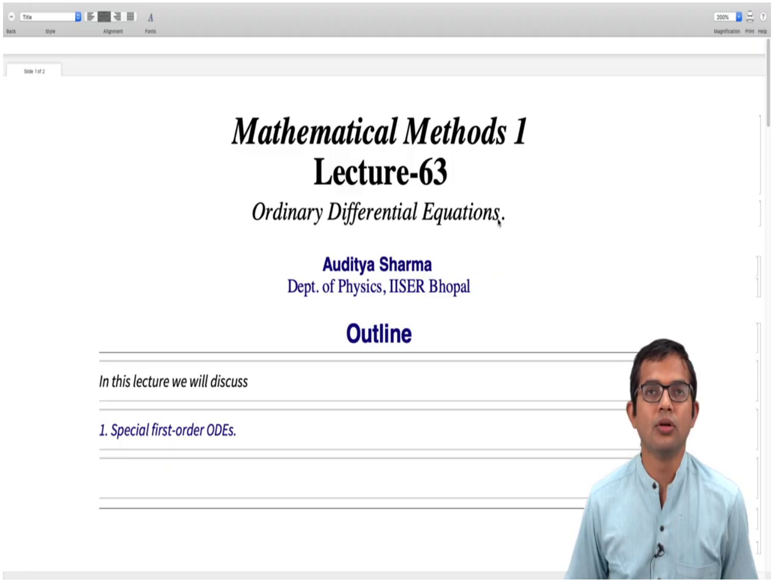

So, before we move on to a study of higher order differential equations, it is useful to look at some special kinds of first-order ODEs right, which can be you know with the help of some transformations cast into a separable form right. So, if it is linear, of course, we know how to do it.

But you know there are more general first-order ODEs which are important which appear in applications and which you know look apparently difficult to solve but there are you know some clever transformations using which they can be, they can be you know made separable right. So, that is the subject for this lecture ok.

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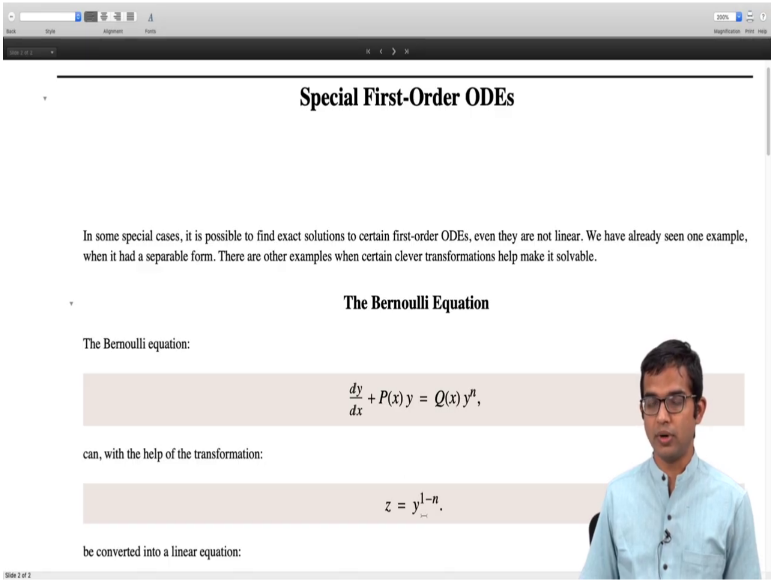

So, you know one class of equations, which are first-order and which are non-linear - they go by the name of the Bernoulli equation. So, the Bernoulli equation is of this kind dy by dx plus some function of x times y is equal to Q of x times y to the n right.

So, clearly because there is y to the n, it is a nonlinear equation - it is first-order because dy by dx is all there is no higher order. But this can be recast into a familiar form if you just make this transformation you write down z is equal to y to the power 1 minus n.

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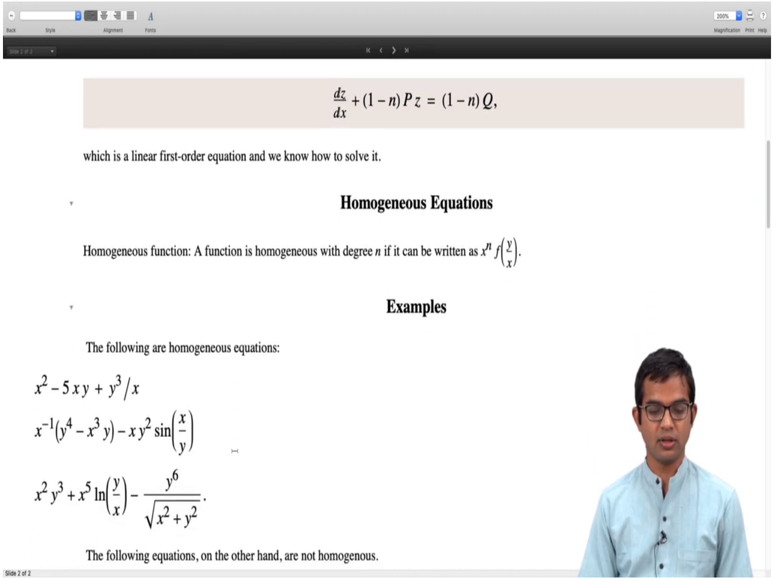

Then you can make it linear in z. So, what happens is, dz by dx is equal 1 minus n times y to the minus n dy by dx right. So, you plug that in, and then you know in place of y to the right. So, you have to substitute throughout. And in place of y, you put it in terms of z and so on, and there will be this cancellation of all these y to the minus n's they cancel out nicely, and then you are left with dz by dx plus 1 minus n times P times z is equal 1 minus n times Q.

So, now, this is a familiar differential equation. It is not only first-order, but it is also linear right in z. And so then we can go ahead and solve it, right. So, this is one class of problems which apparently look difficult, but which can be solved using all the methods that we already are aware of. We spot the form and we make use of this the right substitution.

So, let us look at another class of equations which can also be solved. And so these are called homogeneous equations not to be confused with the term homogeneous we used earlier to refer to you know equations where the right hand side was 0 right, so that is one kind of terminology. But homogeneous equations are also equations involving homogeneous functions of, a function is homogeneous and with degree n if it can be written as x to the n times f of y by x right.

So, basically the function cannot be an arbitrary function of x and y, but there must be a way to treat this as a function of y by x. The powers of y and the powers of x in every term it was conspired in such a way that f of y by x is there is a meaningful way of thinking of this as a function of y by x right.

So, let us look at a few examples. So, if you have something like you know all the you know these equations are all homogeneous equations x squared minus 5 xy, so you have you know the power here is 2 x times y you know 1 plus 1 is again 2, and then y cube by x again it is you know effective power here is also 2.

So, you know one way to see this explicitly is to pull out x squared and so you see that the first time becomes 1, the second term will become minus 5 y by x, and the third time will become y by x the whole cube. So, and indeed you see that this is the, you know function of this form. And the second one is also a homogeneous equation because here you can pull out y to the 4, and then you get x cube by y cube, 1 minus x cube by y cube.

And you know you can think of this as so and then you also have this second term x y squared is there, but which you can write it as y, if you pull out a y cube, then it becomes x by y right. So, the first equation is x, x inverse y to the 4, and then you have the second equation which is y cube times x, x by y. So, y to the 4 divided by x, so y by x.

So, y cube you know you can pull out again out of the entire equation, you have y by x. And here if you pull out y cube you have x by y, right. So, it does not matter whether it appears with x by y or as y by x. You can convince yourself that this indeed works out here as well. And then the third equation also works out. You can check that indeed you can write this equation as say if I pull out x to the power 5 from this equation, so it is y by x the whole cube comes in here.

So, and indeed there is an x to the power 5, which comes in here, so which will go with you know y power 5 divided by x to the bar 5, and then there is an overall y which comes to the denominator and indeed the third equation also works out. So, you have to check this. Sometimes it looks deceptive, but in fact, you can convince yourself with a little bit of effort that it is a homogeneous equation.

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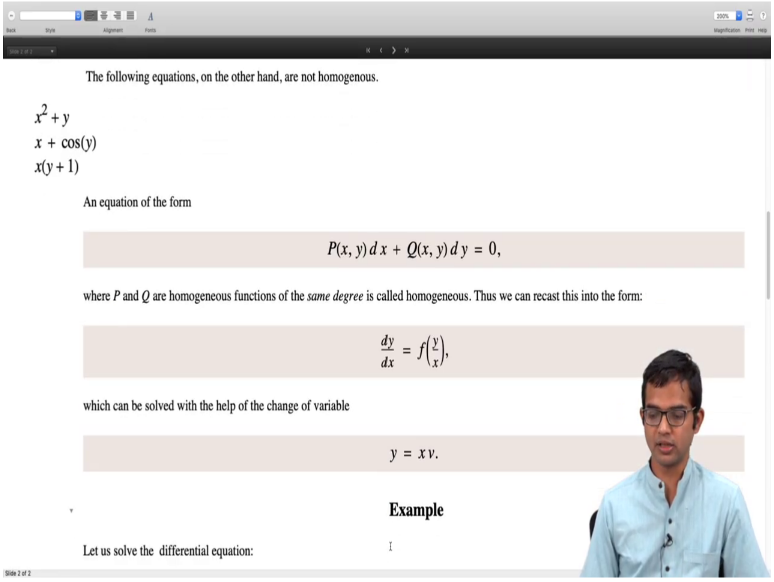

And then to contrast, let us look at a few examples where they are not, they are not homogeneous. So, no matter what you do, you cannot bring it to this form right, x squared plus y is one such example, x plus cos of y is another, x y plus x is another such example right. So, you know homogeneous equations are very special, right.

And when an equation is homogeneous, an equation of this form you say is homogeneous if P you know there are these two functions P of x comma y and Q of x comma y, if both P of x comma y and Q of x comma y are homogeneous functions of the same degree right, so then you say that this whole equation is homogeneous.

So, then it is a very special form, then it turns out that you can write dy by dx as f of y by x. And if this is possible, then there is a way to at least formally solve this differential equation; no matter how crazy this f is right. So, you can just make the substitution y is equal to x times v, and then you can solve for it.

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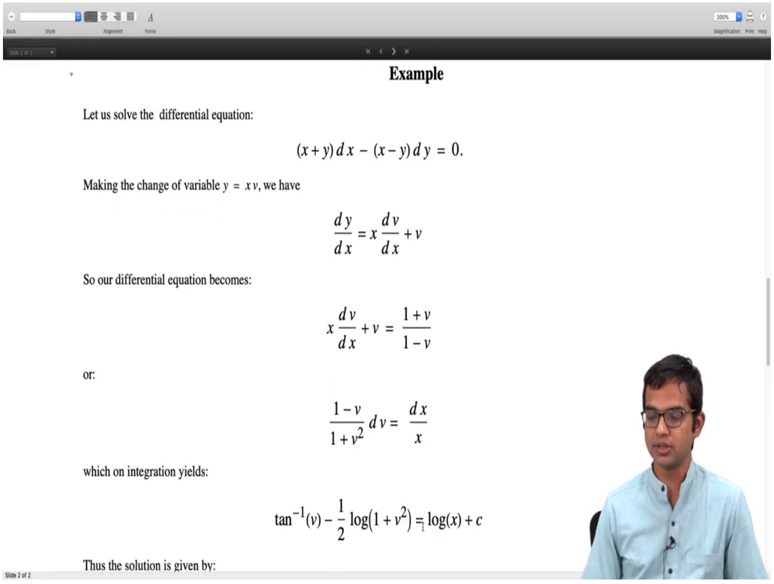

So, let us look at an example to make this concrete. So, if I have x plus y times dx minus x minus y dy equals 0, clearly, this is a homogeneous equation because x plus y and x minus y are both homogeneous functions and both of them have the same degree right. So, the degree is this you know this power n. So, you can immediately see that both of them have the same degree in this case. And so if you make the substitution y equals xv, so then we have dy by dx is equal x times dv by dx plus v.

If we go ahead and substitute for this, we have x dv by dx plus v that is dy by dx. So, dy by dx must be equal to x plus y divided by x minus y, but which can be written as 1 plus v divided by 1 minus v. And then you can just bring all the terms involving v to the left hand side and all the terms involving x to the right hand side, so the x part is very easy if you just get dx by x on the right hand side.

And on the left hand side, you get 1 minus v divided by 1 plus v squared. So, you write it as two separate times 1 over 1 plus v squared minus v over 1 plus v squared the first of this will just integrate to tan inverse of v, and the second one of them will give you minus half log of 1 plus v squared. You can check this and convince yourself that this holds which is equal to log of x plus c.

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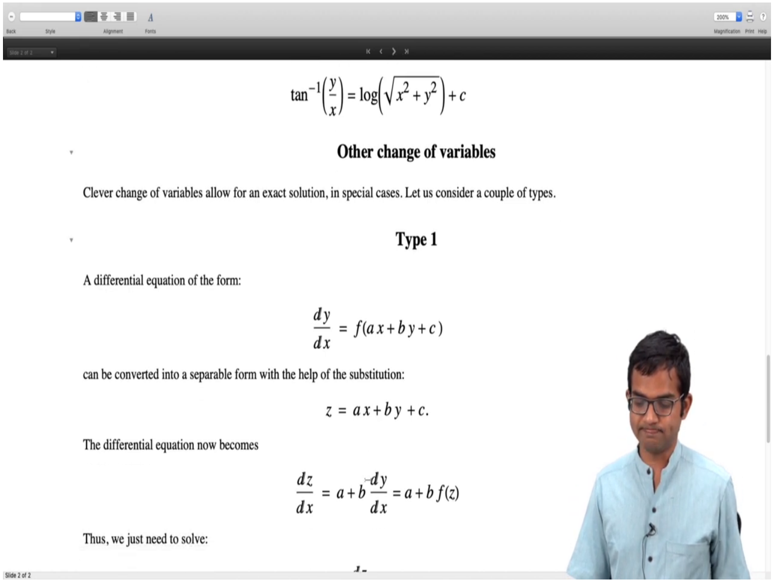

So, the solution is given by. So, now, we have to go back and plug in place of v you have y by x, then you have you know this gives you square root of 1 plus y squared by x squared which you can write it as x squared plus y squared by x squared. So, log x minus log x will cancel with you know minus times minus log x will cancel with this log x on the right hand side.

And so basically the final answer will be just tan inverse of y by x is equal to log of square root of x squared plus y squared plus and overall constant. So, there are other clever changes of variables which are very special to special forms. There are a couple of special cases which we will look at, right. And maybe there will be some homework looking at other kinds of change of variables which are (Refer Time: 10:30).

So, type 1 is if you have a differential equation of the form dy by dx is equal to some function of ax plus by plus c on the right hand side. So, even if the function which appears on the right hand side is a very complicated function, if it is has this integrity right, which is you know this particular form, then you can just make the substitution z is equal to ax plus by plus c, and then dz by dx becomes a plus b times dy by dx which is the same as a plus b times f of z. So, the right hand side you know so your differential equation in z is as dz by dx equal to a plus b times f of c.

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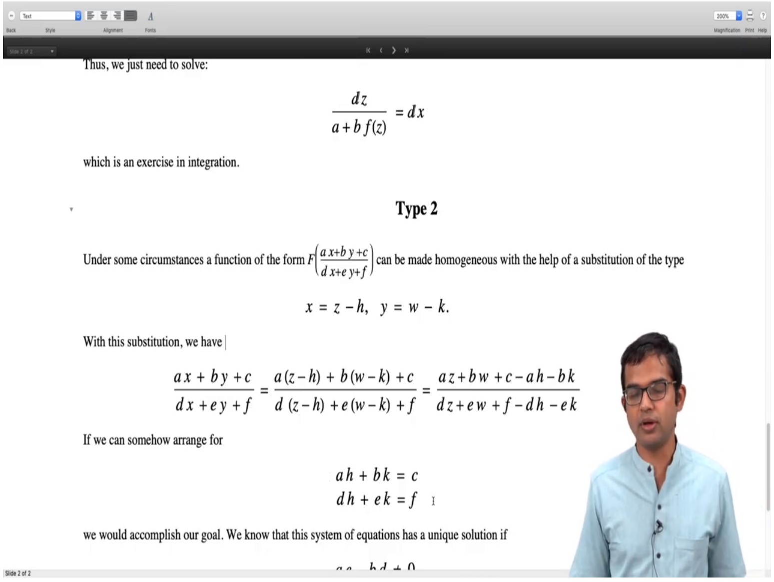

So, you can separate terms, I bring all the terms in z to the left hand side, and all the terms in x to the right hand side which is dx. So, if you are able to do an integral which may not always be possible, integral of dz by a plus b times f of z, then you are done right. And then of course, you have to go back and plug in for z in the final you know final solution and then you have a solution for y itself, which may not be easy to write down in closed form.

But at least there is a formal solution available for differential equations of this form. Now, there is another type where you may have a, you know the right hand side is not this simple function of ax plus by plus c, but it may be a function of this form F of ax plus by plus c divided by dx plus e y plus f.

Now, if this happens, you can actually make this homogeneous with the help of a substitution of this type. Where x in place of x you write z minus h and in place of y you write w minus k, you shift these variables by amount which we will see what that should be. So, we demand that you make the substitution right. So, your ax plus by plus c divided by dx plus ey plus f.

Now, it becomes az plus bw plus c minus ah minus bk over dz plus ew plus f minus d h minus ek. So, we do not want these constant terms if you can somehow make these zero both in the numerator and in the denominator, then we will get a homogeneous form for the right hand side. So, if we can somehow arrange ah plus bk equal to c and dh plus ek equal to f, then we would accomplish our goal.

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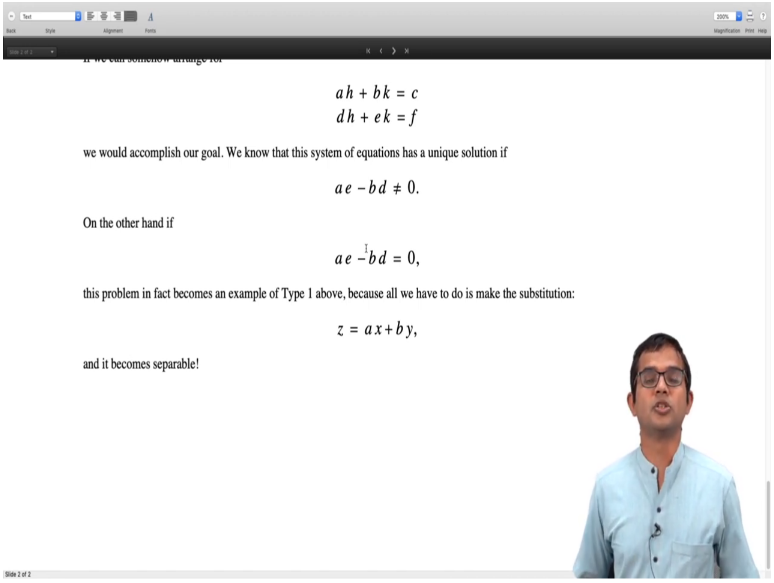

But we know that this is a system of equations in h and k which has a unique solution provided this determinant ae minus bd is nonzero right, so that is the condition. If you can, if ae minus bd is nonzero, then for sure there is this transformation which can make your right hand side homogeneous, and then we know how to solve right. So, we have seen the prescription for that.

But on the other hand, if ae is minus bd is equal to 0, so it is not a problem. If it is, then this problem is actually a type 1 problem right. You see the numerator and denominator. So, what does ae minus bd equals 0 mean? It means that you know when you have something like this, a function of this kind.

So, you have a plus ax plus by in the numerator, but you have dx plus ey in the denominator can be thought of as some eta times ax plus by right. Of course, there is c and f. But basically you can just substitute for ax plus by, you can make the substitution z is equal to ax plus by, and then it is separable formally at least right.

So, we will give you some homework where you know there will be some example problems where you will have to work this out, but the idea is that a differential equation dy by dx is equal to some function of this complicated looking expression can also be solved formally right.

If this determinant ae minus bd is nonzero, then you homogenize using this prescription. On the other hand, if it is 0, then you just make the substitution z is equal to ax plus by and solve it like we did earlier right.

So, the main message from this lecture is that you know when for first-order equations, which have a certain special form, even if it is non-linear there are these nice you know substitutions which may be possible some transformations which can recast this problem into a form from where we know how to solve this problem ok, that is all for this lecture.

Thank you.