Mathematical Methods 1

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Fourier Transforms
Lecture – 54
Fourier Sine and Cosine transforms

Ok. So, we have seen how the notion of a Fourier series for a periodic function can be generalized to the notion of a Fourier transform for a non-periodic function, and how you know the summation of the series expansion now becomes basically an integral right.

So, a Fourier integral function can be written as a Fourier integral right, in terms of the inverse Fourier transform, and then there is a forward Fourier transform which gives you the Fourier transform of the function right. So, very often in practical applications, we encounter you know functions which may be non-periodic right. So, for transforms are pertained to functions which are non-periodic, but they are either even or odd.

So, they have you know well-defined parity. And in such a scenario, you know it is more convenient to work not with the full Fourier transform itself, but define a special kind of Fourier transform which is you know either a sine Fourier transform or a cosine Fourier transform. So, this is the going to be the content of this lecture ok.

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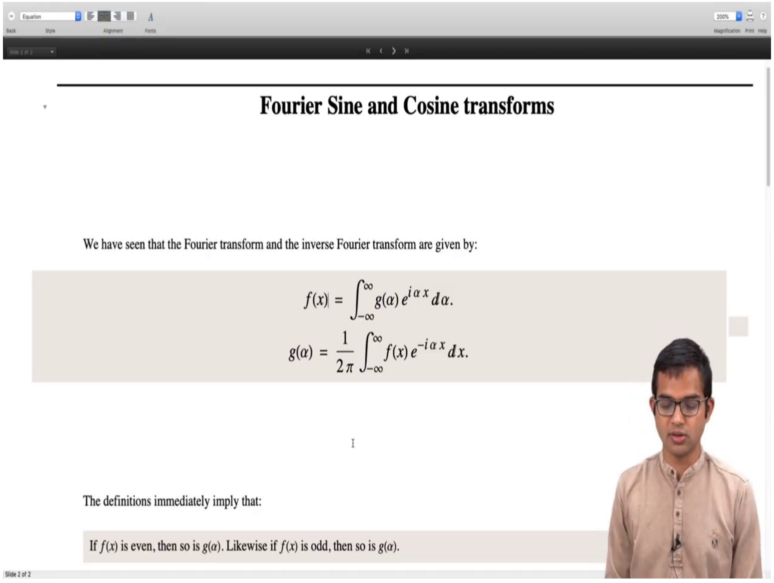

So, we have seen how you know any arbitrary function f of x can be written as an integral minus infinity to plus infinity g of alpha e to the i alpha x d alpha. And then where g of alpha is the Fourier transform of the function which is given by another integral 1 over 2 pi integral minus infinity to plus infinity f of x times e to the minus i alpha x dx right.

So, it is important to mention that you know this factor of 1 over 2 pi which appears you know in the definition of g of alpha is a matter of convention right. So, the rule is that if you put a coefficient c 1 in the first equation and a coefficient c 2 in the second equation, the product of these two coefficients must be 1 over 2 pi.

So, you will see in you know different textbooks, different ways of you know dividing these two factors c 1 and c 2 right, so very common convention is to use 1 over square root of 2 pi in each of these integrals right. So, sometimes you may find 1 over 2 pi in the other in the definition of f of x and so on right.

So, it does not matter as there is nothing sacrosanct about what definition you use. So, the important thing is of course, to be consistent right. So, if you define Fourier transform in a certain way that immediately fixes what the inverse Fourier transform must be ok.

So, let see what happens if f of, so the statement is the following: if f of x is even then so is g of alpha, and if f of x is odd then so is g of alpha. Then this does not care about what factors you ascribe to them right clearly. So, let us see how this follows directly from the definition.

So, if you want to pause the video and work this out for yourself, please feel free to do so. So, the argument is straightforward.

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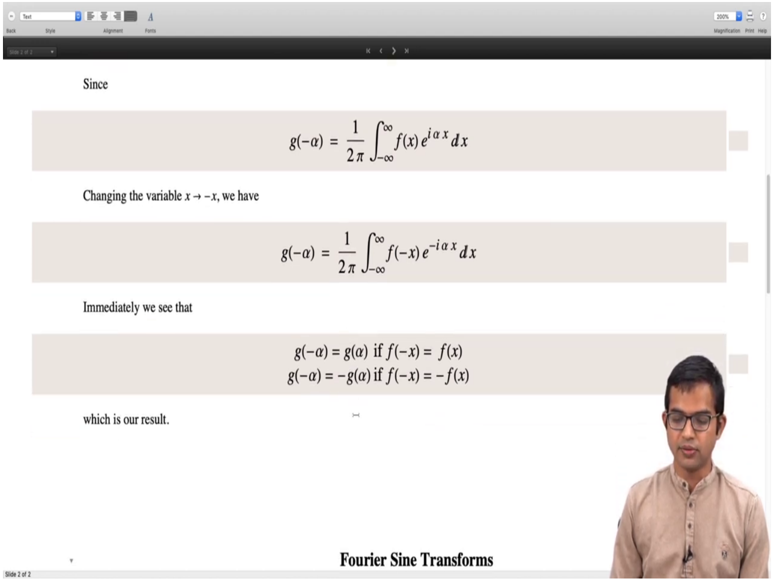

So, we write down g and we have the expression for g of alpha. So, we are able to immediately write down an expression for g of minus alpha. So, you just replace alpha by minus alpha, and then you have 1 over 2 pi integral minus infinity to plus infinity f of x e to the i alpha x dx right.

So, now if you change the variable of you know x to minus x right, so I mean you can use some other dummy variable it does not matter, but it is convenient to just rewrite it in terms of x in the end. So, then x becomes minus x. So, you again this e to the i alpha x will become e to the minus i alpha x, dx will become minus d x, and your limits you know also get changed.

So, this factor of minus 1 which comes from dx and this you know reversal of the limits if you know operate if you do both of them, then you might as well do nothing right. So, if you can just write down g of minus alpha as this expression 1 over 2 pi integral minus infinity to plus infinity f of minus x e to the minus i alpha x dx. From this expression, we immediately conclude that if f of x is odd so is g of alpha, and if f of x is even then so is g of alpha right. So, if f of x is even, then f of minus x is going to be equal to f of x.

So, then of course, you immediately see that g of minus alpha is going to be g of alpha. On the other hand, if f of x gives you a minus sign that will reflect in g of minus alpha as well. So, g of minus alpha will then become minus g of alpha. And therefore, you see that the oddness or the evenness of a function gets reflected in the oddness or evenness respectively of its Fourier transform right.

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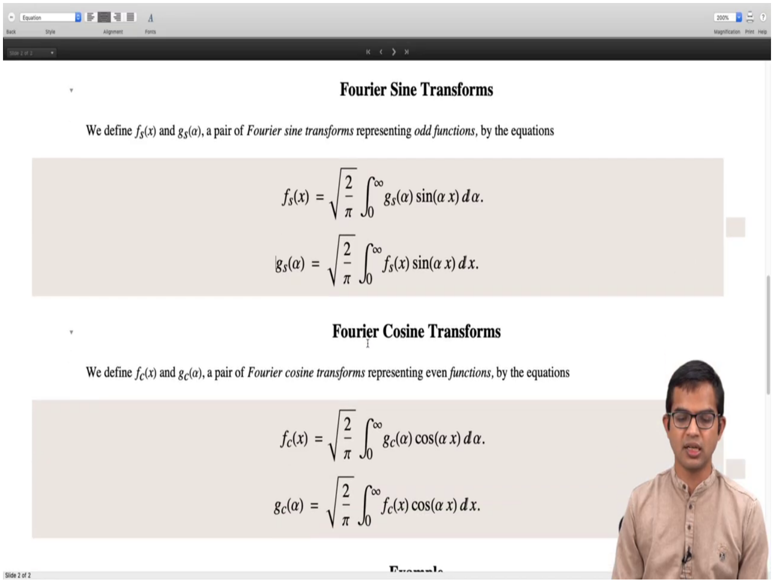

So, because of this, we can define what is called a Fourier sine transform if you have an odd function right. So, if you have an odd function, so you if we have seen that you know all the information for you know pertaining to a function is contained you know in the interval 0 to infinity itself.

You do not have to go from minus infinity to infinity right. It is whatever information is available you know to the right of the origin you know it has full information about for you know all values of x to the left of the origin as well right. So, the definition for right, so here we have used a symmetric way of splitting these factors right.

So, this is just some convention again like I said right at the beginning you have the freedom to pick these coefficients. So, I am just using a symmetric definition in which these coefficients are shared in the you know in the same in an identical manner between the forward transform and the inverse transform.

So, if you, so if you have an odd function right, so you just have to look at this and convince yourself that this is; this makes perfect sense right; you can start with your original Fourier transform as it is right, and then you see that you know the integral from minus infinity to plus infinity can be written as you know it is just the integral from 0 to infinity with appropriate factors right.

So, now, you have, so the definition for the Fourier sine transform is simply given by square root of 2 by pi integral 0 to infinity f s of x sin of alpha x dx. And the inverse Fourier transform gives you the representation for your function in terms of this integral right, so integral from 0 to infinity g s of alpha sin of alpha x d alpha alright.

So, this is you know to be viewed in parallel you know to these expressions. And so you see that instead of your limits going from minus infinity to plus infinity, here it will go from 0 to infinity.

And similarly there is the notion of a Fourier cosine transform. Once again you know even in an even function all the information about the function is contained in the right half of your x-axis, you do not have to go to minus x at all right. So, here it makes more sense to work with cosines right. So, you have, so this sine part is going to cancel right. So, it is only the cosine part which will live.

And so you have 0 to infinity g, so f of f c of x right, the subscript is you know tells you that it is a courier cosine transform is in play and it is an even function. So, you have square root of 2 by pi integral from 0 to infinity g c of alpha cosine of alpha x d alpha. And the Fourier cosine transform itself is defined as g c of alpha is equal to square root of 2 by pi integral from 0 to infinity f c of x cosine of alpha x dx.

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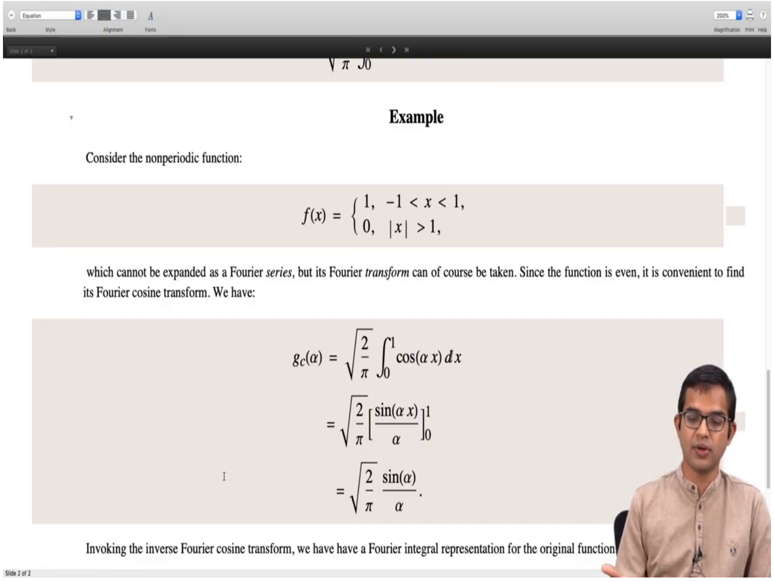

So, Fourier transforms have lots of applications. So, let us just look at one example. And here we will actually use you know work with a function which is even. So, consider this non-periodic function f of x is equal to 1 from minus 1 to 1; and it is 0 if mod x is greater than 1 right.

So, the Fourier transform of this function can be taken; although, you cannot expand it as a Fourier series unless you extend this function right. So, we have seen that in the past. So, we have seen how you know if you are just given a function in some interval, you can assume that it is you know for certain applications it is useful to think of this as a you know periodic function whose you know one period information has been given right.

So, but suppose we do not do that right we can do a Fourier transform of this function. And since the function is even it is convenient to find its Fourier cosine transform. So, now, we have g c of alpha is equal to square root of 2 by pi right like we have seen the definition, so the integral is from 0 to 1 cosine of alpha x dx.

And then you have so that gives you a sin of alpha x divided by alpha and the limits are from 0 to 1 and so only the limit at x equal to 1 survives and so you are left with square root of 2 by pi sineof alpha divided by alpha right. So, if we invoke the inverse Fourier cosine transform right, if g c of alpha is the Fourier cosine transform of f of x, then f of x must be the inverse Fourier cosine transform of g c of alpha.

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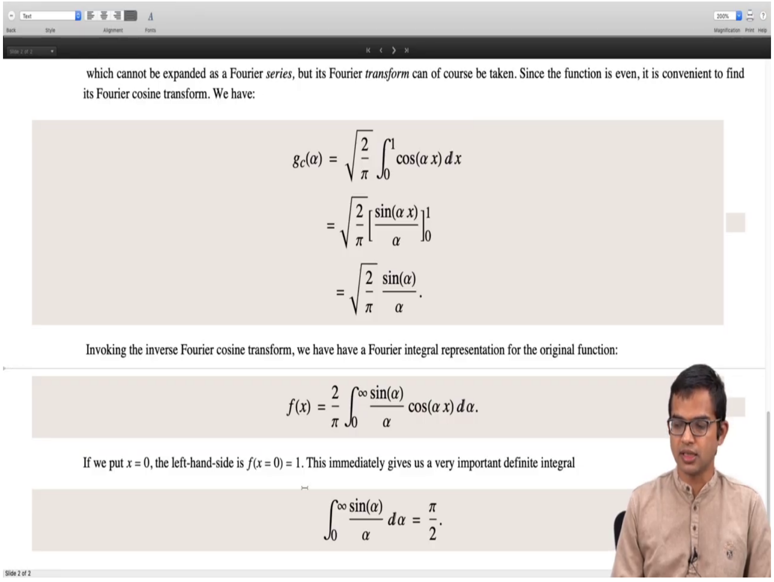

So, which means f of x should be written as this Fourier integral 2 by pi integral 0 to infinity sin of alpha divided by alpha times cosine of alpha x d alpha. And so this result immediately yields us a very useful integral which appears in you know many contexts, and so that is obtained by you know putting x equal to 0.

So, we see that you know you can plot this function and see that at x equal to 0, there is no you know there is nothing weird about this function at x equal to 0, it is super smooth. It is you know well behaved x equal to 0 is continuous in all the nice properties that you can want for your function are satisfied. And so the left surely this integral is going to converge. So, it is going to have the value 1.

So, if we put f of 0 equal to 1 equal to 2 by pi integral 0 to infinity sin alpha by alpha d alpha right, and then we bring this factor of pi by 2 to the left hand side or you rewrite the whole thing. So, basically what we have managed to show is this integral 0 to infinity sin alpha by alpha d alpha is equal to pi by 2. So, this is a very useful result. This can be proved in other ways right, but we see the power of the Fourier transform right.

So, this example is an exercise in illustrating how Fourier transforms also can do magic tricks for us. So, we saw how Fourier series can sometimes give us infinite series and there are these magic results which we have shown some examples of, and there will be homework along these lines as well. But so the point is that Fourier transforms can also give us these kinds of magic results and just one simple example of which has been illustrated in this lecture. That is all for this lecture.

Thank you.