Mathematical Methods 1
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Fourier Series
Lecture – 52
Parseval's theorem

Ok. So, in this lecture we will look at something called Parseval's theorem right which is a way of you know working out a quantity based on all the coefficients, right. So, we have seen that if you have a periodic function which can be written as a Fourier series you get a whole bunch of coefficients.

So, there is a convenient number that you can form by collecting all these coefficients which is related to a very nice form of the function and this has some very beautiful applications, one of which we will also look at in this lecture, ok.

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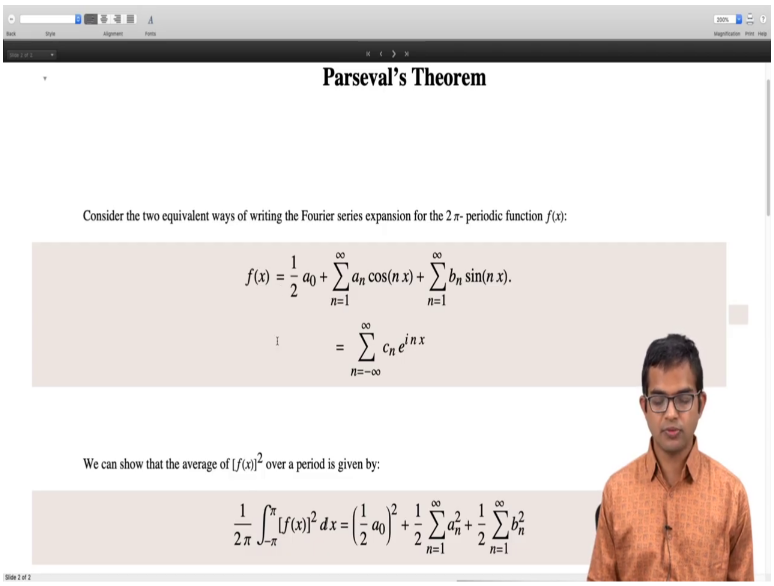

So, there are these two equivalent waves ways of expanding a function of period 2 pi right one is in terms of cosines and sines and the other in terms of exponentials, right. So, we know how to do this. And so, now let us look at what happens to this function f of x, right. It is a periodic function.

So, if you want to consider the average of the square of this function, right. So, you know we have seen how electrical engineers are often interested in the average value of the square of certain functions right. So, like AC current for example, right, it is helpful to measure the average of the square.

So, if you measure the average of f of x squared over a period then it is given by you can check this. So, what would happen is so, the left hand side is just integral minus pi to pi f of x the whole square d x 1 over 2 pi, right since you are have that is the total interval; now, the right hand side when you I am just giving you the answer, but you can check this explicitly.

So, what happens is you have these you know every term you take a product of this entire sum with itself now. So, it is going to give you all these terms which are you know squares of itself right that is one term that you get, but you also get these cross terms, right. So, we have seen that it is profitable to think of this as an expansion of a vector in an orthonormal basis right, that is what Fourier series really is right.

So, if you take the square and then you know you integrate in a period, that is basically like taking the inner product of these different basis vectors with each other. So, we have seen that they are orthonormal or orthogonal for sure right, normalization you have to work out. So, and we have left it open, but basically any two you know distinct basis vectors their inner product is 0.

So, therefore, none of these cross terms will contribute when you do this integral and half a naught the whole square will just give you half a naught squared when you average over a period and then we have seen that cosine squared an x right, so, the average of cosine squared or the average of sin squared both of them are just half. So, therefore, you will get half summation over n a n squared and another half summation over n b n squared, right.

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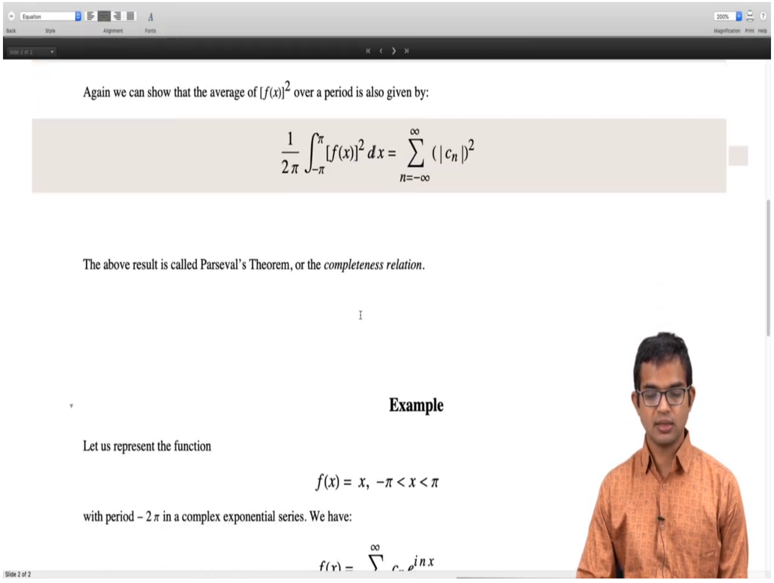

So, again we can show that the average of f of x squared if you are working it out in terms of the other coefficients of a pertaining to the exponentials then you. So, in general these c n s are complex numbers right, as we have seen. So, here you will get mod of c n square, right.

So, this is also something you can explicitly verify by you know first of all arguing that the cross terms the integrals pertaining to the cross terms will vanish and the integrals involving just the diagonal terms. You know you can show that it is exactly this quantity it is going to give you mod c n square summed over all n going all the way from minus infinity to plus infinity.

So, this relation is called Parseval's theorem or the completeness relation right. So, it is called completeness because you know there is this hint of you know these basis vectors which form a complete set they form a basis. So, there is this completeness which is satisfied by all these cosines and sines or all the exponentials, right.

Even if one of them is missing it is not going to be possible for you to expand every vector in the space in terms of all these exponentials right you know if even one of them is missing right. So, each of these basis vectors is an essential ingredient right the coefficient along each of these and so in fact, that is what you know allows it to be called as a basis, right.

So, it is a very you know compact set in the sense that each of them brings in a certain amount of information which the others are not able to give on their own and also there is no redundancy, right. So, everybody is valuable and there is and everybody is essential and there are no unnecessary members in the set.

So, that is why we have seen that for finite dimensional spaces you know all bases had exactly the same number of elements. So, here it is an infinite dimensional space. So, the number of you know elements in your basis is infinite, but you know cosines and sines are you know they form together they form a basis meaning that there is no redundancy here, right.

So, you know when it comes to finding the size of the basis one may have to be careful, but the point here is that all the cosines are essential. Even if one of them is missing you will not be able to expand an arbitrary vector, arbitrary function in terms of a series like this if even one of these frequencies is missing every one of them is key, right.

So, that is the reason behind the name completeness relation for this theorem. Let us look at a beautiful application of Parseval's theorem. So, let us consider this example: f of x equal to x from minus pi to plus pi with period 2 pi - we write it as a complex exponential series. So, we want to work out these coefficients c n such that f of x is this infinite series.

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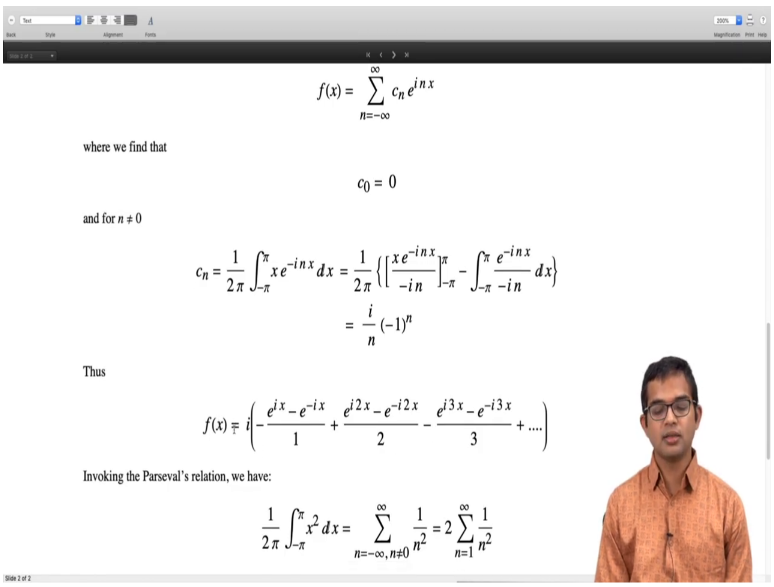

So, c is not immediately we can check that it is just 0 because it is going to be just the average of this function right it is going to go to 0 evidently. So, and when n is not equal to 0 to find c n we must do 1 over 2 pi integral minus pi to pi x times e to the minus inx dx. So, we can do this by parts.

So, you have u d dv which is u v minus v du. So, you can convince yourself that the second integral is just 0, right, because this is the n the integral of cosine and sine in you know in an interval which is an integral multiple of the period. So, it does not matter what n is, this integral is going to vanish.

And, then if you work out the algebra carefully you will see that this will give you i over n times minus 1 to the n, right. So, if n is negative then because you have n sitting here in the denominator you are going to have one more you are going to have a one more negative sign, right. So, on the one hand you get this minus 1 to the n which you know oscillates when you have an even and odd n, but also if your n itself is negative that will give you another sign, right.

So, there are these two kinds of signs you have to be careful with. So, if you I work this out carefully you know the series expansion for f of x is i times minus i to the i x minus yeah e to the i x minus e to the minus i x divided by 1 plus e to the i 2x minus e to the minus i 2x divided by 2 minus e to the i 3x minus e to the minus i 3x divided by 3 so on, right.

So, you can see immediately that these are all going to be only sinusoidals. No cosines are allowed in this expansion right that is because this is an odd function right. So, it is more convenient to work with these complex exponentials for the application of Parseval's relation, right.

So, that is why I have written it like this. Now, if we bring in Parseval's relation right. So, we have to find the integral of f of x square dx you know and divide by 2 pi in the interval minus pi to pi. So, f of x is the same as x which is x square dx, the summation is equal to summation over all n from minus infinity to plus infinity except n equal to 0, right.

So, we see that we have i over n. So, mod of n times minus 1 to the n, mod of this c n squared is going to be just 1 over n square right. So, i does not contribute, mod of i squared will just become 1 minus 1 to the power n to the whole squared again, will just become 1.

And, yeah of course, we have to omit n equal to 0 we have seen that c naught is equal to 0 separately we have seen that. So, this is the same as saying 2 times summation over n you know going from 1 to infinity only the positive values of n allowed 1 over n square right, there is symmetry in this.

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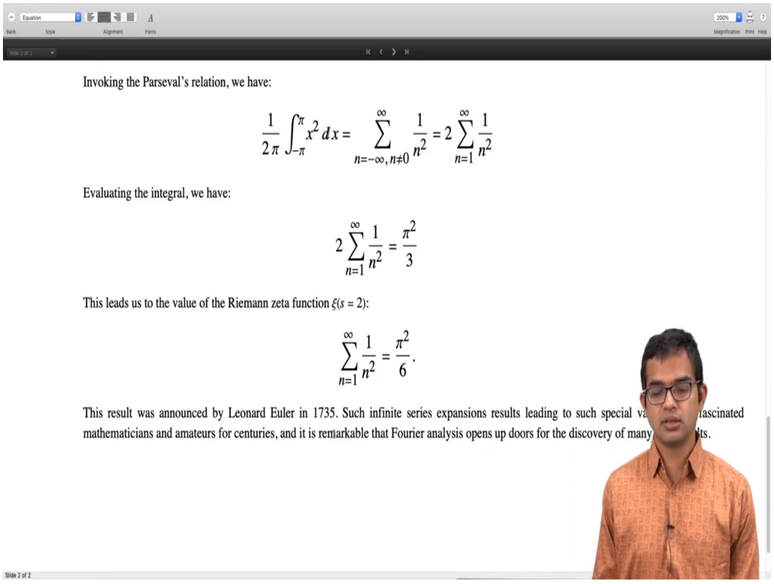

So, what we have is the result that 2 times summation over 1 over n squared is pi squared over 3 or in fact, what we have found is the value of the Riemann zeta function at s equal to 2 right. So, you might be familiar with the Riemann zeta function which is defined as summation over n going from 1 to infinity of 1 over n to the s, right.

So, s is the variable. But, if you put s equal to 2 then that is this Riemann zeta function - that is what we have here and we have managed to show that this is equal to pi squared by 6. So, this is a very beautiful result. So, you know in the 21st century, perhaps it is not something that is unknown or it is not something which is unfamiliar, a lot of people are aware of this result.

But, it still carries a certain you know magic value even by 21st century standards right. And it was first discovered and advertised by Leonard Euler in 1735, right. So, he used other methods. So, there are plenty of different ways of obtaining this, right. So, we have managed to get this using Parseval's relation within the Fourier series analysis.

So, there are many independent ways of doing this sum which are more elementary than others, but this is a beautiful result right. So, we have already seen another series - the Leibniz series result which was also sort of like a magic trick which got us there. And, in fact, there is a whole bunch of this kind of series which one can extract using Parseval's relation or you know other methods Dirichlet conditions you know.

So, these are all things which we can play and we can in fact, come up with our own functions with periodic functions and look at what happens you know with Parseval's relation of, we can use this as a way to try and find our own series; and, then these are probably most likely going to be series that are already known which we can verify, ok. So, on that note, that is all for this lecture.

Thank you.