Mathematical Methods 1
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Fourier Series
Lecture – 48
Complex Form of Fourier series

So, we have seen how a periodic function with period 2 pi can be written as a Fourier series in terms of sines and cosines. And we have also looked at the Dirichlet conditions for convergence of such a function.

So, in this lecture we will show that there is an equivalent exponentials approach right. So, instead of sines and cosines we could express a periodic function in terms of exponentials, right. So, the 2 are completely equivalent, but in some contexts it is more convenient to work with this exponential form that we will lay down in this lecture right.

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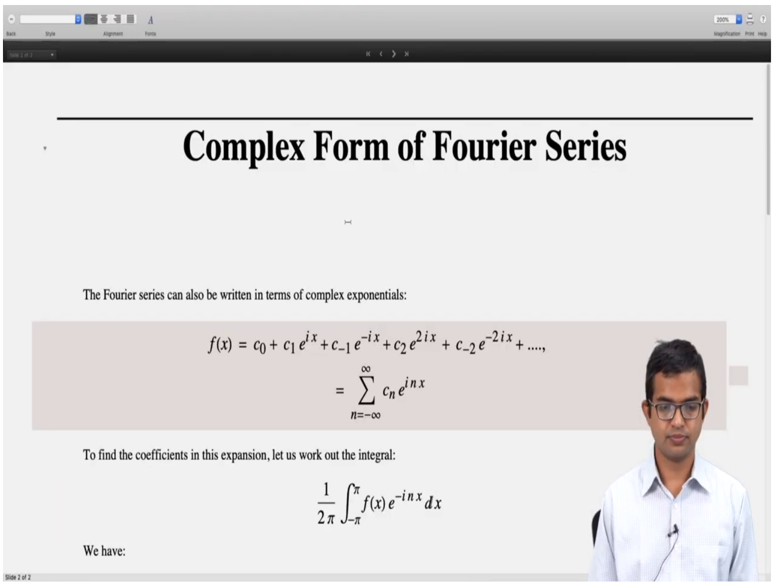

So, you are given a periodic function with period 2 pi. So, we are going to write this f of x as c naught plus c 1 plus e to the i x plus c minus 1 e to the minus i x plus c 2 e to the 2ix plus c minus 2 e to the minus 2ix so on, right. So, you have also these you know now you allow for this index to take negative values as well, right.

So, in other words we are writing this as a summation where n goes now takes all integer values from minus infinity to plus infinity c n e to the i n x, right. So, to find these coefficients c n right so, let us work this integral on right. So, you are doing this integral 1 over 2 pi integral minus pi to pi f of x times e to the minus i n x dx.

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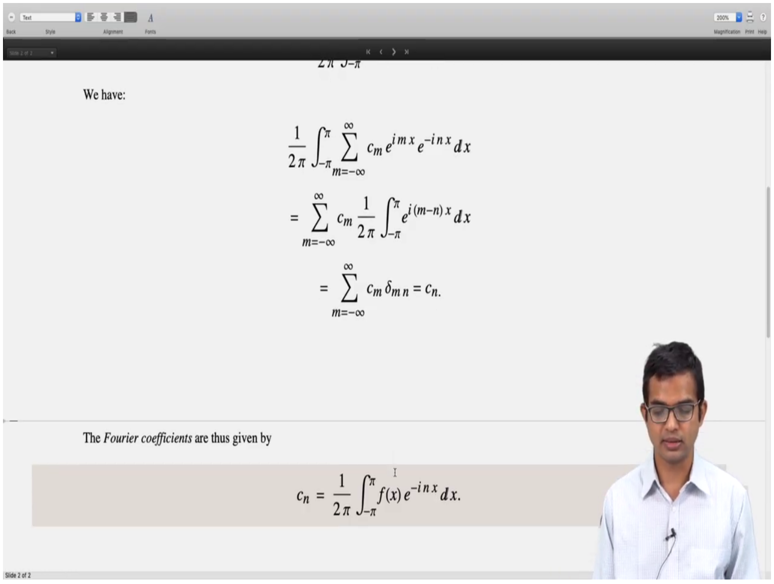

So, now we have 1 over 2 pi integral minus pi to pi summation over m instead of n I am using m it is a dummy index c m e to the i m x, then I have e to the minus nx dx. So, I am going to exchange sums and integral right. So, which it turns out is possible right.

So, because we are working with a function which satisfies Dirichlet conditions and you know the convergence theorems have been worked out we are not proving any of them right. So, this is a legitimate thing to do because of the type of functions we are working with right.

So, let us see this argument through right if you know allow for this exchange of summation and integral. So, this integral is just going to give you a 2 pi times delta m n right, if m equal to n then you get a 2 pi if m is not equal to n right this integral is going to be 0 right. So, you can check this.

And, then so, what it does is basically it is going to pick out just one of these coefficients. So, in fact, this integral is going to pick out for you the one coefficient that you are interested in. So, in fact, this is a prescription for identifying any of the Fourier coefficients.

So, this is our rule. If you are given a function f of x and you want which is periodic with period 2 pi and you want to express this as a Fourier series involving these complex exponentials, then the coefficients are given by just this simple expression. So, the advantage with this formulation is that it is just one single expression to remember.

C n although you know you can argue for it and directly work it out whether you are doing it with the sine cosine approach or with the c n if you have understood the logic of this. C n is equal 1 over 2 pi integral minus pi to plus pi f of x e to the minus i n x dx, right. So, this is you know nothing spectacularly new has happened here right it is just completely equivalent to the sine and cosine approach.

We could have also just started with the sine cosine series and in place of whenever you have sine you replace it with you know e to the sine x can be writ10 as e to the x minus e to the minus x divided by 2 pi and cos x can be writ10 as e to the x plus e to the minus x divided by 2, right. So, if you do this in a in the sine cosine Fourier series you will anyway get back the same result, right.

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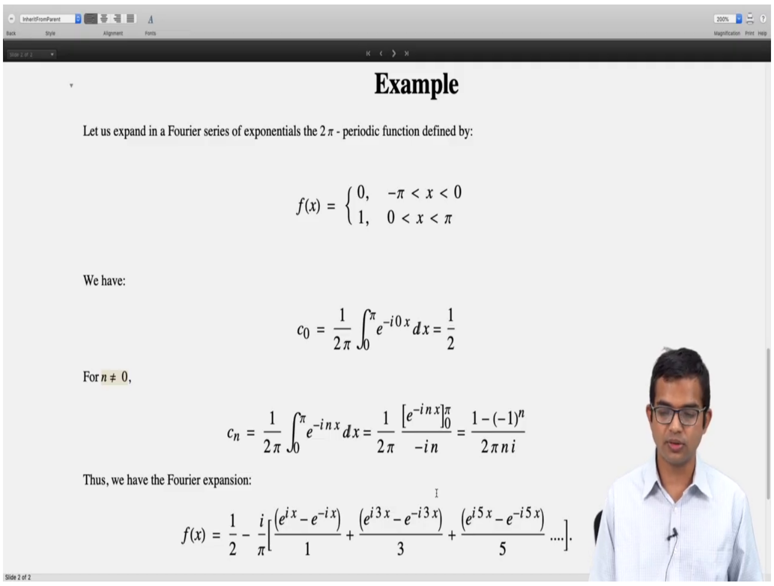

So, let us look at an example right the same example which we have already seen, but using this prescription. So, let us expand in a Fourier series of x complex exponential this 2 pi periodic function f of x equal to 0 in the interval minus pi to 0 and 1 in the interval 0 to pi.

So, we always have to be careful with this c 0 term, the coefficient corresponding to the DC part so to speak. Usually needs to be treated separately right you cannot blindly use this formula and and think that you have the full answer right you always use some special caution to work out the dc coefficient. So, c naught although the formula is the same right, but it is easy to slip in this evaluation of these coefficients, if you do not make a distinction for c 0.

So, c 0 is 1 over 2 pi times just one in the interval 0 to pi, right. So, the other interval does not matter minus pi to 0 because you have 0 here for the function and so, you just get a half for c naught. And, for n greater than or equal to 1 so in fact, also negative n as well, right. So, I should say mod n greater than or equal to 1. So, let me just say n not equal to 0. So, for n n not equal to 0 c n is given by this expression which you can check is just 1 minus 1 to the whole power n divided by 2 pi n i.

So, thus we have this Fourier expansion. If you collect all these terms carefully you know you can pull out a pi in the denominator and there is an I in the denominator which you can shift it to the numerator and you have a minus sign you know the factor of 2 also is there and then that is also it cancels with you know one of the factors inside and then so, if you if you carefully look at all the factors involved you can write down this Fourier expansion, right.

So, you can go back to the previous discussion about you know writing this in terms of sines and cosines and convince yourself that indeed we had only sine, right. We had only sines and if you look at the way I am grouping these terms you are going to get only sines.

So, this expansion in terms of exponentials is completely equivalent to the expansion in terms of cosines and sines, but in certain contexts it is more convenient to work with you know this formulation and there are other contexts where that is more helpful, right.

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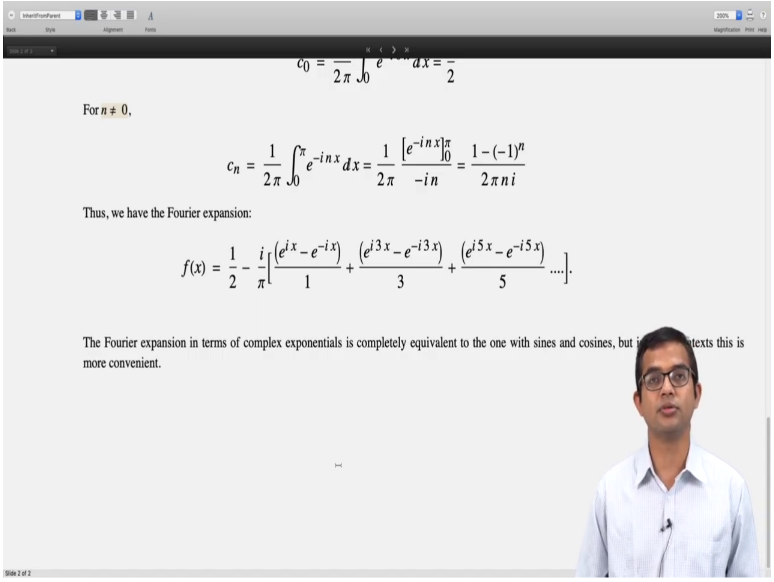

So, that is all for this lecture. We will come back to exponential formulation for other kinds of problems later on, but you know this lecture is just to point out that there is an alternate completely equivalent formulation for Fourier series.

Thank you.