Mathematical Methods 1

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Linear Algebra
Lecture – 37
A class of invertible matrices

Ok. So, we saw how; when you go from one basis to another representations of operators undergo a similarity transformation, right. So, there are some special kinds of similarity transformations which can take your representations into a very special form, right. So, in order to go there, we look at a class of invertible matrices, right.

In this lecture, we look at a class of useful invertible matrices and which we will come in handy in our discussion of this transformation, a special kind of basis transformation which we will discuss ahead, ok.

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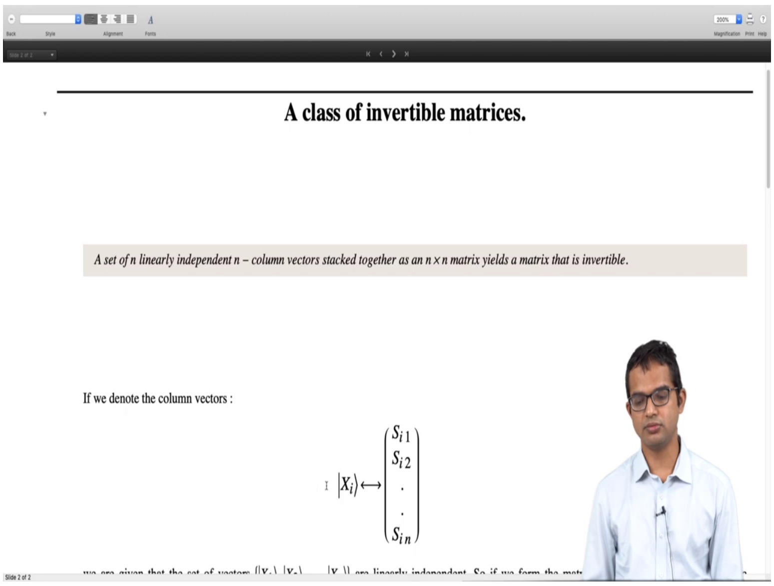

So, if you take some set of linearly independent n column vectors and just stake them together and form an n by n matrix, right; so, we will argue that such a matrix is going to be invertible, right.

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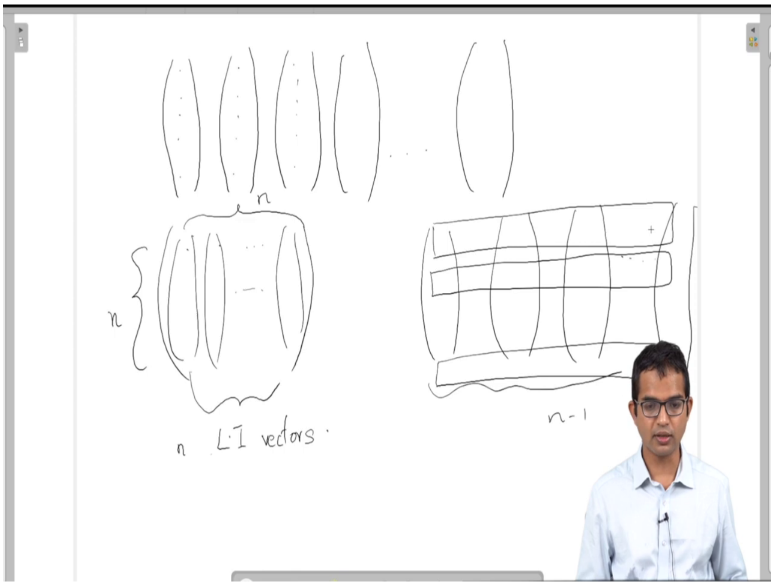

What I am saying is I start with some n column vector. So, I have a bunch of numbers here. So, then I chose another vector which is linearly independent, meaning it is not just some multiple of this. So, I have another vector you know made up of n elements, then I have a third vector made up of again n elements which is you know each of these 3 vectors linearly independent. The fourth one there is (Refer Time: 01:53) that is also linearly independent, so on.

If I put together exactly n of n such vectors and then form a you know an n by n matrix. So, I have you know some coefficients all the way up to you know n, exactly I have n columns and exactly n rows. But the key point is that each of these vectors is linearly independent, so linearly independent, I have n linearly independent vectors, right.

So, if I am able to do this then I am guaranteed that such a matrix is invertible, right that is the statement, ok. So, yeah; so, I am showing it again. So, I denote these vectors. I have a vector X, X 1 which is I will call these elements you know S 11, S 12, so on up to S 1 n. In general, the i-th vector will be S i 1, S i 2 all the way up to S i n, right.

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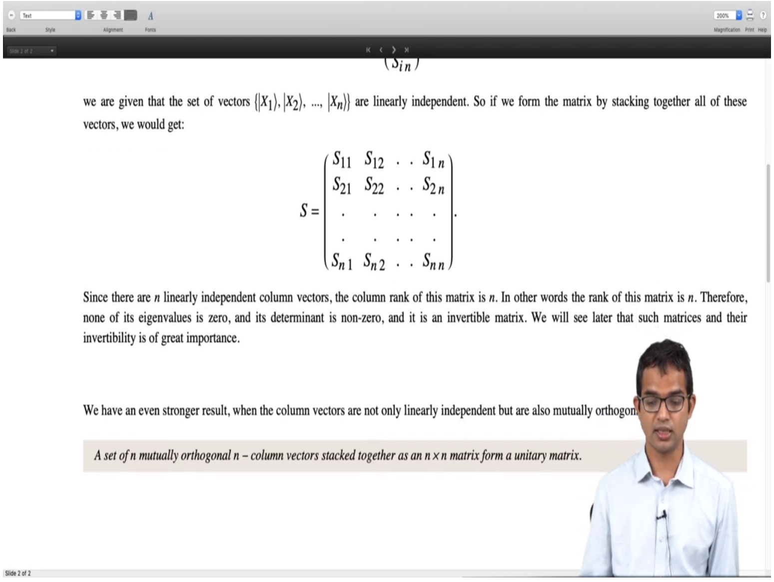

So, we have given these vectors X 1, X 2 all the way up to X n and what we are given is that all of them are linearly independent and so, we form this matrix S 11, S 12 all the way up to S 1 n; S 21, S 22 all the way up to S 2 n and so on

Now, the assertion is that this matrix S is invertible, right. So, how do we argue for this? It follows from our understanding of the rank of this matrix, right. So, we know that the column rank of such any matrix is the same as the row rank of any matrix. So, if you look at you know there are n column vectors which are all linearly independent and we have seen that the column rank is the same as the number of linearly independent column vectors.

So, there are n of these. Therefore, this matrix is the row rank is the same as the column rank and it is the rank of the matrix is n every you know row and every rank carries information that cannot be you know got from other, rights. There is no way to you know take linear combinations of rows or columns of this and then reduce it to a you know effectively a lower number of rows or lower number of columns.

Every column is essential, right, that is what it means. We have seen this and you know we have played with these ideas in the past, right. So, you can go and review some of these results if you are confused.

But the point is that since there are n linearly independent vectors it is a matrix of rank n and that means that none of the eigenvalues of this matrix can be 0, because if it were 0 then it would be in that its determinant is 0. And you know if its determinant 0 matrix then it cannot be you know full rank matrix it has to then you have to look for sub matrices of dimension n minus 1 and so on, right. That is one prescription we gave.

Another way to think about it is that if an eigenvalue is 0 then at least you know one of these rows is sort of redundant - that is what is meant by some eigenvalue, one or more eigenvalues being 0. And for sure if none of the eigenvalues is 0, the determinant is nonzero; it is an invertible matrix.

So, as you have also seen you know the inverse of a matrix will contain a factor of 1 over the determinant. If the determinant is 0 then it is not invertible, but if the determinant is nonzero then 1 over determinant of this matrix is not a problem.

So, why do we care about this? So, you will see that you know this has an important application. You know with this, we are interested in forming matrices you know which are obtained by staking together a bunch of n column vectors. Now, there is a special kind of linear dependent n column vector, right. If you had staked together not just n linearly independent n column vectors, but also those who are mutually orthogonal, right.

Like for example, when we are doing Gram-Schmidt orthogonalization process that is what we are doing right, you start with the vector then you get another vector, and then you have to do some tricks to make the second vector orthogonal to the first one; you start and orthonormalize them in fact, right. And then you take a third one which is orthogonal to both the first and the second and also you normalize it, right and so on.

So, if you do that then we have a stronger result. In fact, if you have you take a set of n mutually orthogonal n column vectors and stack them together to form an n by n matrix, then this resulting matrix is not only invertible, but in fact, it is going to be unitary, you can read off its inverse, right.

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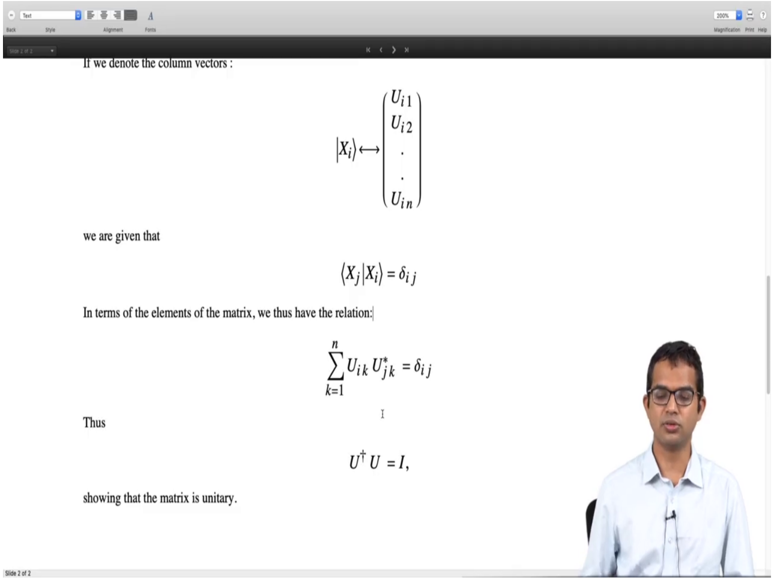

So, it is kind of a remarkable and sort of a very beautiful result that this happens, right. So, you know you are working with some Gram-Schmidt orthogonalization process and then you find one vector, you find a second vector, third vector, and so on. You know it is not such a surprise that you can make a bunch of vectors which are you know this guy is orthogonal to this or you know normalize orthogonal to this and then this guy is normal to the first two and so on, right.

But you have let us say you have worked this way and you have come up to n minus 1 and then you put in one more vector and then suddenly you have not just one set of orthogonal you know orthonormal set of vectors, but in fact, you have suddenly created one more. Because it is like a somehow, I will jigsaw puzzle which fix together in such a way that it actually makes this matrix unitary.

So, what it does is it is going to make all these rows orthonormal. You get one more basis for free, in some sense it is a magic the result which happens which is you know you create one you know complete set of orthonormal vectors, you just put them all together in a matrix and since the matrix is unitary, so you know that U dagger U is the same as U U dagger.

So, whether you work with the vectors along this direction or you know the row vectors, they also have the same property. So, until you know you get to the n minus 1th vector; you do not have it is not even obvious that there is this type of a structure happening forming along the rows, but there is this in-built mechanism. So, the moment you put this final vector n all of these click and then you have a you know the same property operating along the rows as well, right.

So, it is like I have some magic trick which happens and in fact, you can exploit this to even find the last vector. You know found n minus 1 of these vectors you do not even have to directly find a final vector such that it is orthonormal to all of these. You can just try to fill out each of these elements along the rows such that each vector is normalized, right. So, that is you can exploit this, ok. So, that is the remark I want to make about this.

So, how do we see this? Right. I mean I have already sort of jumped ahead and said a few things about this, but it is not a very difficult argument for why this is a unitary matrix, right. It just comes from the orthonormality property. So, if we denote this column vector as X i you know in this manner then we are given that X j X i is delta ij that is the orthonormality property.

Now, in terms of the elements of this matrix you can expand this and write it out, summation over k U i k, U j k star, right. The star comes because you have taken the bra vector, right. So, you have to do this complex conjugation and then this is equal to delta ij, but if you look at this it is the same as the condition for the unitary matrix.

So, U dagger U is equal to I, showing that it is a unitary matrix, right. So, U dagger U equal to I is equal to U U dagger and so that is where you get this you know the constraint between the row vectors also obviously, automatically it clicks together in some jigsaw puzzle fashion, ok.

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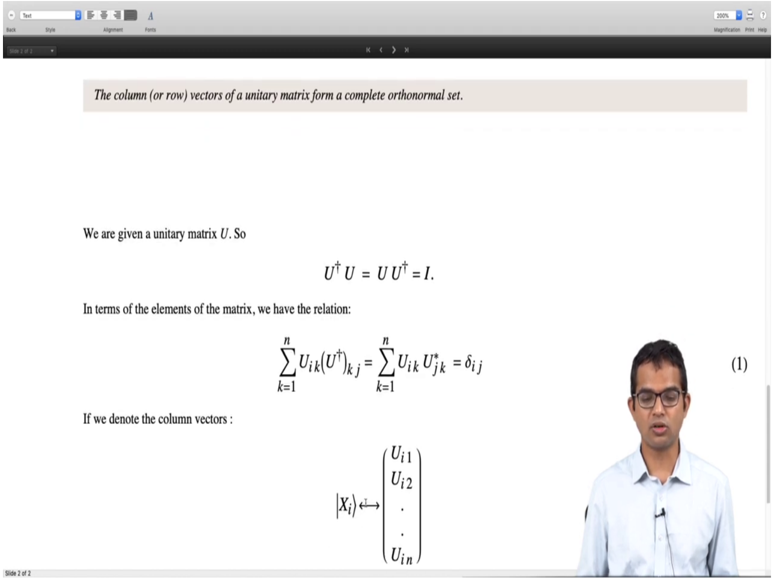

So, the converse result is also of course true. If you started with a unitary matrix then for sure the column or row vectors form a complete orthonormal set, right. So, this is in-built into a unitary matrix.

So, you have given a unitary matrix, so this holds, and therefore, you can write out you know the elements of the matrix you have this relation and this relation is the same as saying you know if you denote these as column vectors, then you know this relation is the same as saying you know X j and X i are orthonormal to each other.

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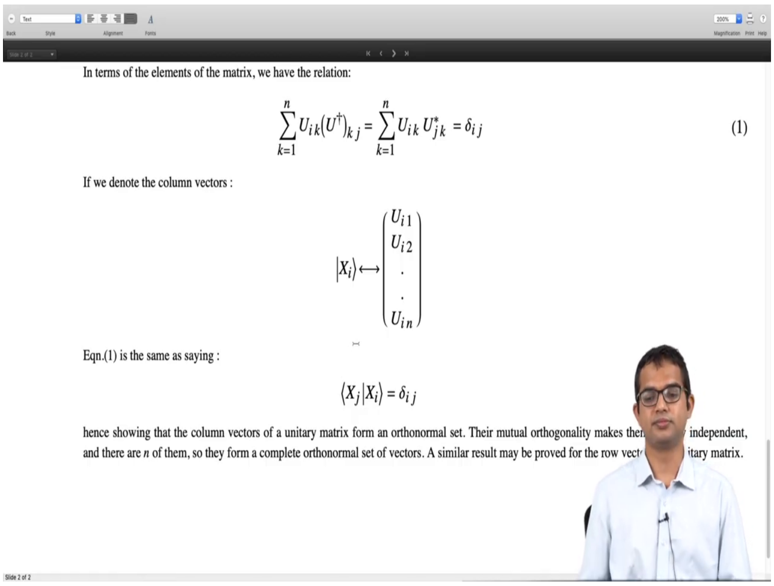

So, explicitly the condition of unitarity immediately implies that you have a bunch of vectors which form an orthonormal set, right. You could start with a unitary matrix and then pull out all the vectors, the row vectors or the column vectors and say that you have a complete orthonormal set or you could start with an orthonormal set of vectors like we do often with Gram-Schmidt orthonormalization process and then you can create a unitary matrix, right.

So, I invite you to use this property and play with the Gram-Schmidt examples that you have already worked with. You know take some three vectors and you know they do not have to be orthonormal, but linearly independent vectors and then orthonormalization using Gram-Schmidt and then you know start filling in the columns, and then you see that magically they will all you know click in such a way that the row vectors also have this property. So, we will see in the next lecture that you know this has a very important and useful consequence.

Thank you.