Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Linear Algebra
Lecture – 36
Transformation of Basis

So, we have looked at the concept of a basis and we are also aware that the basis is not unique. You have typically infinitely many different kinds of a type basis that you can construct for a given vector space although the number of elements in every basis has to be the same, that is the dimension of the space and we have been concentrating on finite dimensional vector spaces. And so in this lecture we look at what happens when you know go from one basis to another, right.

Within a basis we have seen that all operators have representations matrix representations inside you know every basis and so, when you transform when you change basis the representation is also going to undergo a change. So, vectors would change from one basis to another how they look and likewise the representation for operators also will change and that is what we are going to look at in this lecture ok.

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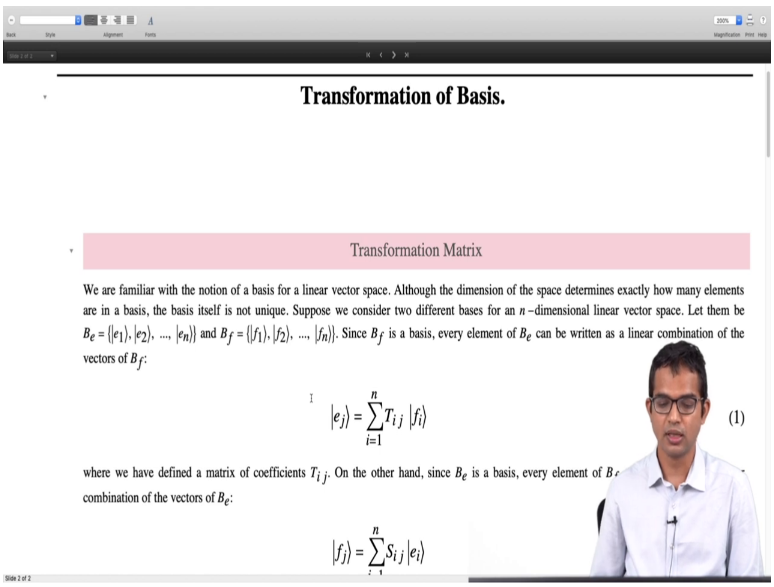

So, suppose you have a basis B e I am calling it B e and you have another basis B f right. So, the first basis has the elements e 1, e 2 all the way up to e n and the second basis has the elements f 1, f 2 all the way up to f n, right. So, necessarily both of them have exactly the same number of elements that is n right because the dimension of the basis of the vector space is n, right. So, each of these bases will have n elements.

Now, at this point I am not demanding that either of these bases must be orthonormal right, that is the special kind of basis we will come to a little bit later. So, let us start very generally. We have two generic bases. So, since B f is a basis, every element of B e right I mean every element of B e is a vector which belongs to that space and so, it can be expanded as a linear combination of the vectors of B f.

So, if we do this you have you know e j can be written as summation over T i j f i summation over i right you can find some coefficients T i j right for every for the j th vector in this basis you can find the coefficients T i j and i goes all the way from 1 to n. Now, so, where we have defined a matrix of coefficients T i j on the other hand B e is also basis and every element of B f can also be expanded as a linear combination of the vectors of B e, right.

So, f j itself can be written in terms of some other set of matrix coefficients some other set of coefficients S i j right you sum over e i sum over i S i j e i right. So, you will always be able to find these coefficients such that you know every vector in one basis can be represented in terms of the basis vectors of the other basis. Now, if I start from equation you know 2 f j is this, but then in place of e i, I will plug back in you know the expression from equation 1, right.

So, then this will give us a you know constraint for these two matrices. So, I have introduced this matrix of coefficients T i j and another matrix of coefficients S i j, but they are you know intimately related. That is what we are going to extract from doing this exercise. So, f j is summation over i S i j, but in place of e i, I will put in summation over k right. It is the dummy index I can choose k because I am working with i here.

So, T k i f k and then I combine you know I exchange these summations. So, instead of summing i first I will put k first and then I will club these two summation over i S i j T k i is and then there is a f k as well, right. So, this is possible only if. So, yeah, so it is more convenient to actually rearrange this and write this as summation over i T k i S i j right.

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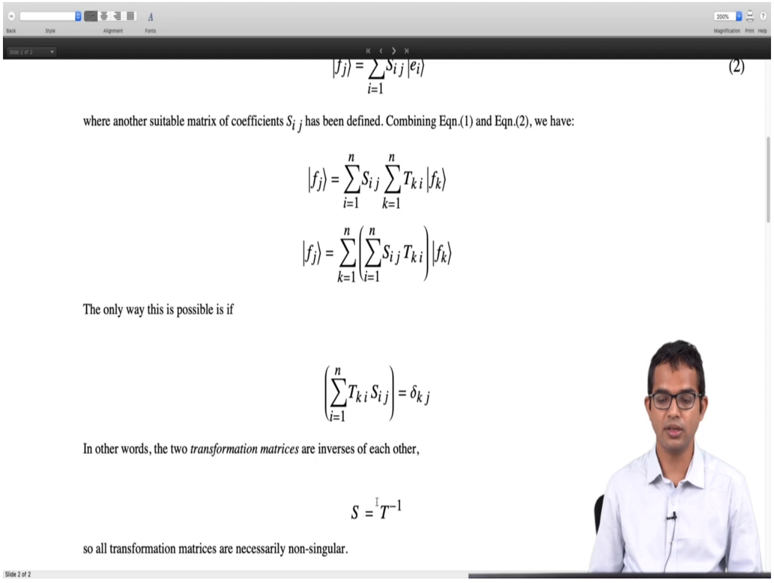

So, then you see that this is in the form of the product of two matrices if I think of T and S as two matrices then their product right must satisfy this condition it is going to be a diagonal matrix and in fact, the all the elements in the diagonal must be 1. Only then will you get back f j right, you have a sum over k, but all these coefficients except when k is equal to j must be 0 and when k equal to j it must be 1.

So, the compact way to represent this, is to say that the summation is now this Kronecker delta k j, alright. Another way of saying this is that T times S the matrix T times the matrix S is equal to the identity matrix or equivalently S and T are inverses of each other right and so.

So, that is what you know is the connection between these two transformation matrices right you go from one basis to the other and there is a you know this basis transformation necessarily must be invertible, right. You cannot have a non-invertible transformation. If you go from one basis to another then there is a way to get back to the original.

So, there is no loss of information in going from one basis to the other right that is what is meant by it being a change of basis ok. So, necessarily both these transformation matrices are non singular right, if it is a singular matrix it is not invertible as we know ok.

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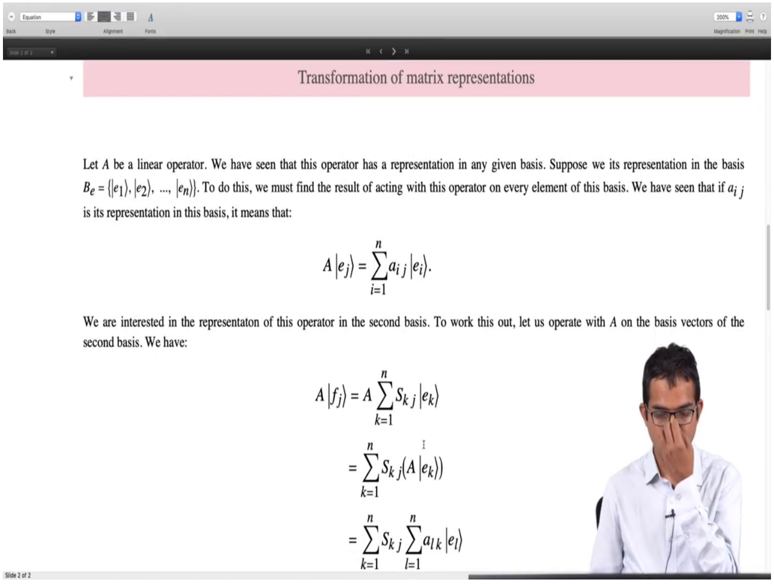

So, what happens to matrix representations and how do they transform right? So, you know we have looked at the basis vectors. So, in general you know vectors also undergo transformations, but so do operators, the representations of operators, right. Operator is an abstract quantity; nothing happens to an operator if you change basis, but the representation of the operator is different in different basis right.

So, let us say you start with B e. So, you have e 1 all the way up to e n and then you know to find out what you know what this operator looks like in this basis, you must find out what it does to each of the elements of this basis, right. So, like we discussed a few lectures ago, right. So, we saw that A acting on e j is summation over a i j e i. So, you explicitly work out what the operator a does to each of the vectors in this basis and then you have.

So, basically you need to find out all these coefficients a i j right. So, if you; so, this is the representation for this operator in this basis and likewise you know there is a representation for the same operator in a different basis, right, the other that is the other basis which I have called B f.

Now, to work this out let us see how we can work out you know what this operator does to the elements of the second basis in terms of you know this transformation matrix and the matrix elements in the first basis of the linear operator, representation of the operator in the first basis we want to connect it to the representation of the same operator in the second basis ok.

So, in other words what we are interested in is finding out what A does to f j. So, what A does to f j is the same as what A does to summation over k S k j e k because we have written f j as summation over you know this i S i j e i you know in place of I am putting k it is a dummy index it gets summed over. So, I have in place of f j I put you know this summation and then it is a linear operator so, it goes right through.

So, I have summation over k S k j you know the operator A acting on e k. Now, but I know what happens to this vector e k when A acts on it because I have written this out in terms of the representation in the first basis. So, let me go and fill that information in.

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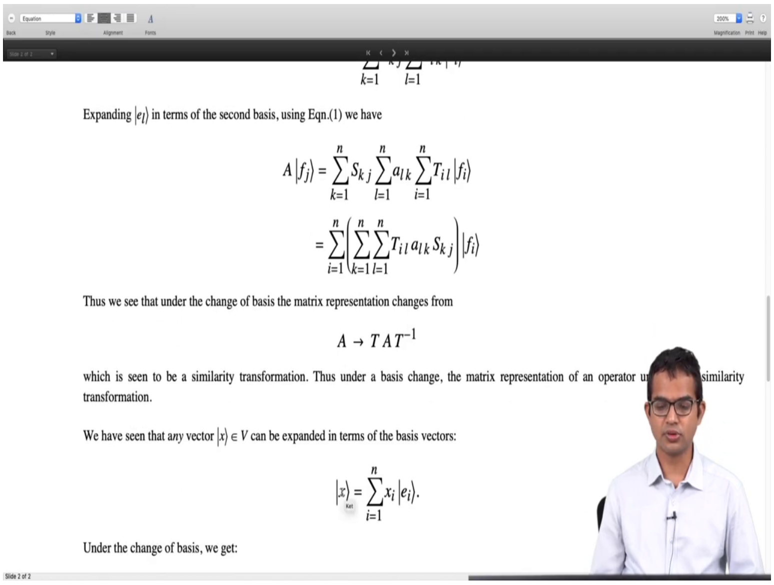

So, I have summation over k S k j summation over l right again it is convenient to use another index other than i here. So, summation over l a l k e l right. So, I am looking at what happens to e k when A acts on e k and so, I can plug this in here. Now you expand e l in terms of the second basis using equation 1, right; e l itself can be written in terms of you know with the help of T i j's you can write it in terms of the basis f i B f, right.

So, then you have one more summation coming in. So, A acting on f j is actually summation over k S k j summation over l a l k summation over I T i l f i, right. So, you should cross check and convince yourself that this is indeed you know I have used the right symbols everywhere.

So, now I can regroup these summations in this way. I will bring the summation over i write at the beginning then I have a summation over k summation over l and then it is convenient to rearrange these coefficients in this way T i l a l k S k j which gives it a very suggestive form right and then finally, you have a summation over f i ok.

So, you see that you know this stuff inside the brackets is really the product of three matrices, right. So, what is happening is. So, we started with the representation a i j, but now you can think you can see that this is the new representation for the same operator A, right. We are interested in what A does to f j right. It must be written in terms of the same basis that will give you a set of coefficients which will give you another matrix right. You can call it a i j tilde if you want right.

So, to avoid clutter I am going to just go to matrix notations. So, you notice that I have T i l a l k if you want you can start with these two and you know multiply these two matrices and then you go ahead and multiply these two matrices.

And, you should convince yourself that really what is going on here is nothing but you know the multiplication of the matrices T, a and S, but S is the same as T inverse as we have seen it has to be right because these two are you know transformations from one basis to the to another and back. So, therefore, it has to be T inverse.

So, all that is happening is your matrix representation is getting sandwiched between these two you know matrices one in the forward direction and the other in the backward direction. So, now you get TAT inverse, which is the similarity transformation as we have seen, right.

Whenever you sandwich a matrix between you know some other matrix and its inverse like here, then it is a similarity transformation, right. So, thus the matrix representation of an operator undergoes a similarity transformation, right.

So, likewise we could have also worked out what happens to any vector x right. So, if you are starting with some vector x. So, you can expand in this basis you get a bunch of coefficients x i which we have seen can be thought of as a column vector, right.

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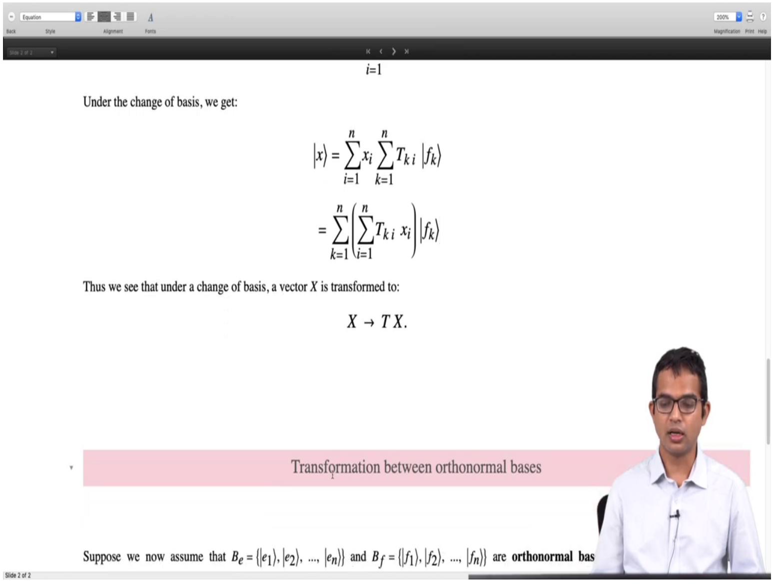

So, under the change of basis we have to just replace e i with you know summation over T k i f k right we are interested in what it looks like in this basis. So, and then we have to regroup these terms and I have a summation over k you know this stuff acting on f k. So, this stuff is the representation of this vector.

So, the vector is the same right, it is an abstract quantity right, if you wish you know this can be as different as you please as long as you know it satisfies the basic properties of vectors, when the representation is a bunch of numbers right, these x i's.

Now, those x i's will have a transformation if you go to a different basis it will look different. Thus we see that under a change of basis a vector is transformed to T times X, hence you should look at this and convince yourself that indeed that all that is happening is you know this vector going from X to TX.

So, representation for an operator you know has this change and the representation for a vector undergoes this change when you do a change of basis right. So, now we look at what happens under a special kind of a basis change or rather when you have two special kinds of basis involved right. We have seen that it is convenient to work with an orthonormal basis, right.

So, when you have an orthonormal basis you get some extra constraints in built right which you know which makes also the transformation between these two such orthonormal basis also come with some special extra properties that is what we will look at, right. So, first of all we have this completeness expansion completeness relation right in each of the orthonormal basis.

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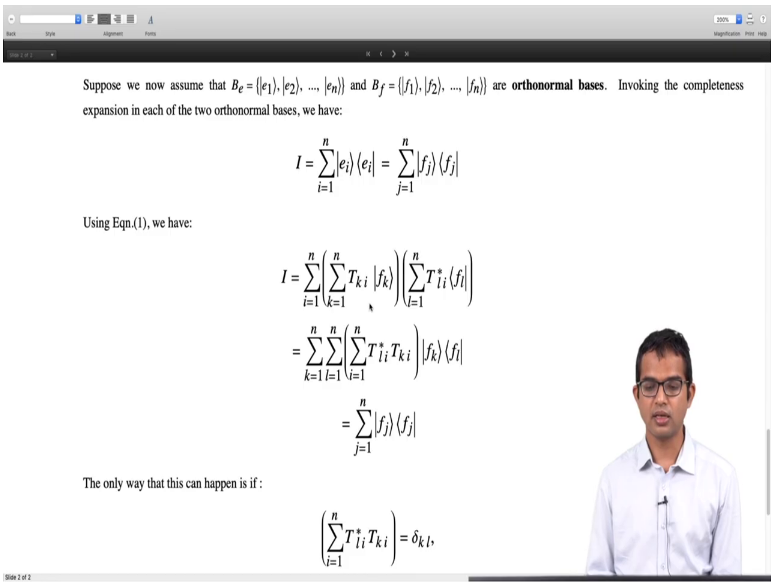

So, identity can be written as summation over i e i e i, but identity can also be written as summation over j f j f j. If you wish you know this is like this is the representation of your identity operator, right and it looks basically the same in any orthonormal basis, right that is what this result mean and we will exploit this to get some more constraints here for you know when you make a transformation between orthonormal basis.

So, see I is summation over i e i e i, but e i in place of e i I will putting this you know there is this way to go from you know right e i in terms of f k so I plug in this here and then I will use a different dummy index. l here, I have a k here. So, the power vectors instead of you T l i you get T l i star right the conjugative has to be taken.

These are coefficients and then if I group them altogether and then I say when I have summation over k summation over l summation over i I bring in right into the sent or then I have summation over I T l i star T k i f k f l. But, this must finally, equal f f j and f j this matrix representation for the identity operator is going has got to be diagonal no matter which orthonormal basis you are implied. So, that means, this has got to be f j f j and that the only way that can happen is if you get a delta function here, right.

So, this quantity must go to delta k l and then you know. So, the double sum will become a single sum and then you get only diagonal terms, right. So, that is what you know that is the condition that is imposed upon these transformation matrices.

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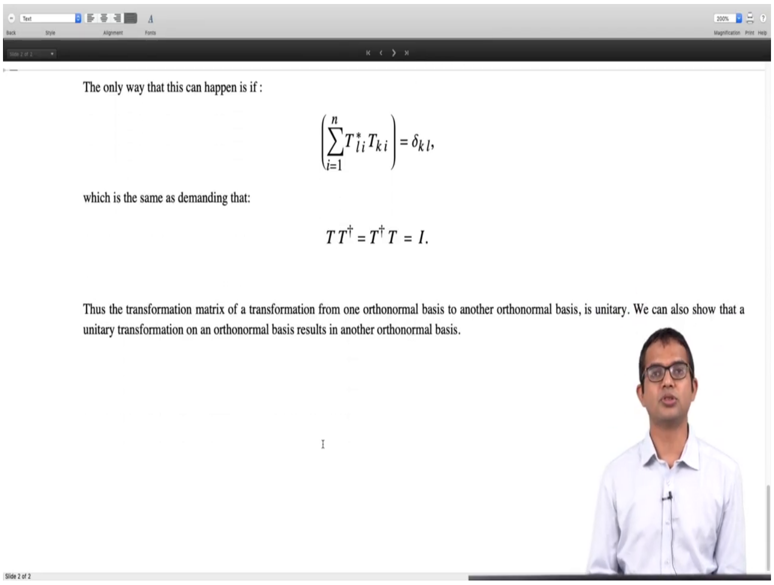

But, if you look at this carefully you know this is the same thing as saying that T times T dagger is equal to T dagger times T is equal to I, right. This is just the same as the condition for this T to be a unitary matrix, right. So, a transformation that takes you from one orthonormal basis to another orthonormal basis is unitary right. So, that is our result.

So, for a unitary operator or unitary matrix, we know that the inverse is the same as the dagger itself; so, T inverse is equal to T dagger. So, the similarity transformation right which you know operators undergo when you change basis. So, doing TAT inverse is the same as saying it is TAT dagger for a unitary transformation ok. So, that is all for this lecture.

Thank you.