Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

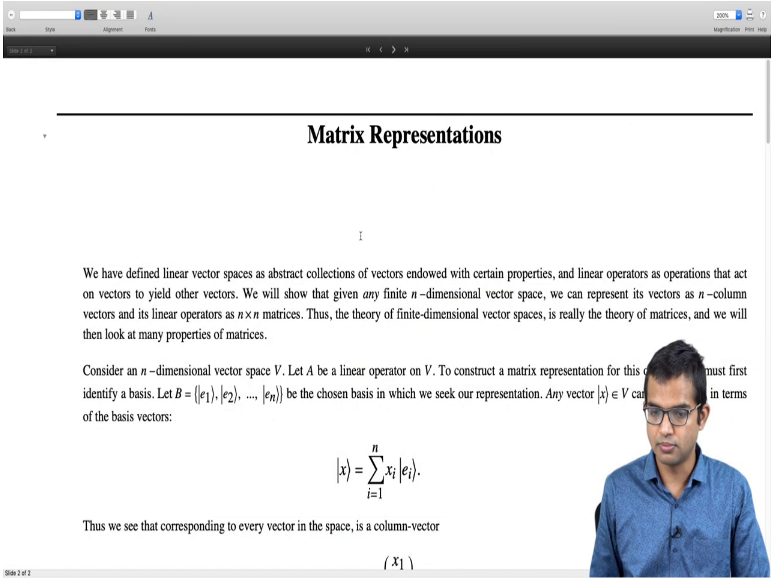
Linear Algebra
Lecture – 32
Matrix representations

Ok. So we have spent a lot of time emphasizing how linear vector spaces contain these abstract objects called vectors and also, there are other abstract objects which are called operators. And you know so the idea here in introducing this abstraction was you know to see how there is some common ground between many different, apparently different kinds of you know objects are really the same in the sense that they have this property that they are a vector right.

And you know there is a certain common ground, where all these apparently different objects can be treated in the same framework right. That is the whole point of linear the discussion of linear vector spaces and so on right. So, today in this lecture, we will show how you know operators can be thought of really as matrices right.

So, if you go to a certain basis right, you can represent an operator no matter how abstract it is and think of it as a matrix. And likewise a vector right, which could be whatever you know it originally is, can be thought of as a column vector right and that is the subject of this lecture right.

(Refer Slide Time: 01:58)



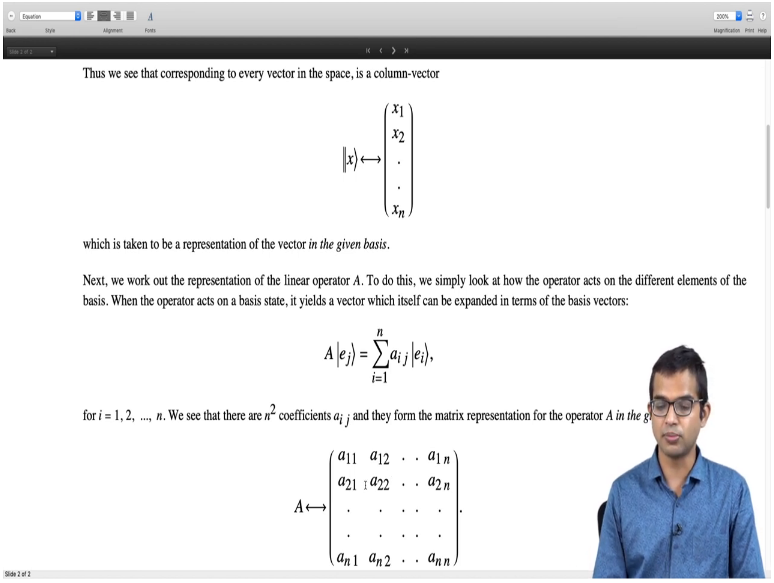
And, so, what we will manage to argue based on this lecture is you know that basically the theory of finite dimensional linear vector spaces is really the theory of matrices. If you understand everything about matrices, you have a full understanding of arbitrary n-dimensional vector spaces right because of this connection that we are about to make to representations right.

So, let us consider an n-dimensional vector space V right and let A be some linear operator on V. Now, you want to construct a matrix representation for this operator. But in order to construct a representation for an operator, we must first choose a basis right. The fact that it is a vector space means that there is a basis, in fact, there are lots of basis in general right; infinitely many basis.

But let us just pick one basis and at this point, we are not demanding that this must be an orthonormal basis. Although later on, we will argue that an orthonormal basis gives you some extra convenience right. So, let us say you have a basis e 1, e 2 all the way up to e n; it is an n-dimensional space so there are going to be n vectors in this basis and so, you know the meaning of a basis is any vector in this space can be expanded in terms of this basis vector right.

So, consider some arbitrary vector x which is an element of the space and then, we will be able to expand this in terms of these coefficients right. And in terms of these vectors x i, you will be able to find n coefficients x i such that you will be able to write x as summation over i x i e i.

(Refer Slide Time: 03:44)



Now, we can think of this set of coefficients itself you know and form a column vector like this alright. So, now, this is just the regular you know column vector that we are familiar with. Although, these abstract vectors x you know could be for example some functions in some function space or it could be some other kind of, it could be even matrices right we have seen, you know complex numbers can form vector spaces you know all kinds of things can form vector spaces.

But here, you know if you just tag together all these coefficients, you know this is just our good old standard column vector right. And so, the point of the you know the view that we are taking here is that do not worry about the abstract nature of this x at all, but just deal with the column vector alright.

So, you can think of this vector as really just this column vector in this basis right. So, that is the key point. So, this is your representation for your vector as a column vector in this basis. And so, you can do something like this for an arbitrary linear operator as well right.

What is a linear operator? A linear operator is you know we have seen that a linear operator is what a linear operator does right. What does it do to various vectors? It is supposed to take vectors and give you other vectors right. So, let us look at what this linear operator does to every single element of some basis right. We have already chosen a basis. So, if you are able to find out what this does to the various basis elements, we have defined this operator right.

So, let us take this operator A and act upon some basis vector e j. Now, this is another vector in the same space. So, it also has an expansion in terms of the same basis vectors. So, you will be able to expand this as summation over some set of coefficients. I have to tag you know both indices i and j; j is a dummy; so, i is a dummy index because it gets summed over; but there is an index j on the left hand side, so that remains.

So, I have a i j e i right. So, this is you know what this operator does to this vector e j is you know represented in terms of this expansion. Now, you know clearly for every j, you will have a different you know n different coefficients are required; i goes all the way from 1 to n and there are n of these. So, in total there are going to be n squared coefficients right; a i j.

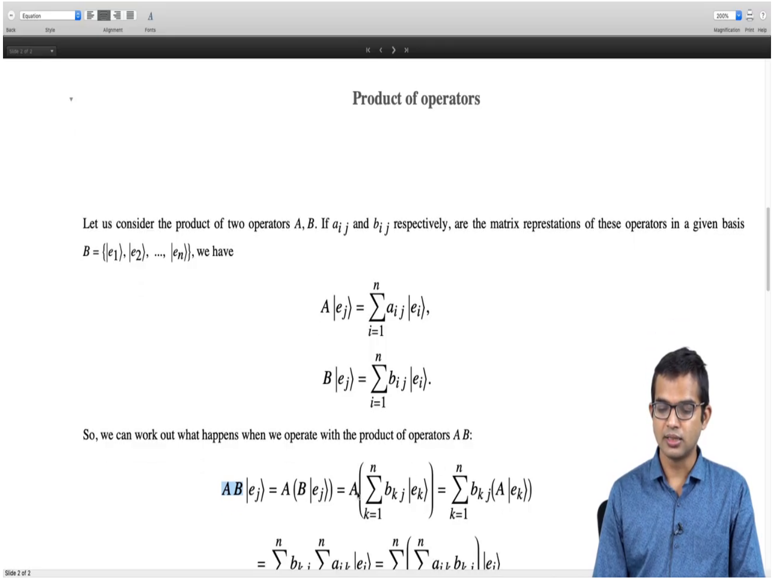
Now, this forms a matrix and this is called the matrix representation for this operator. So, we have seen that if you know everything that this operator, you know what this operator does to every one of the elements of some basis, then we know everything about this operator right.

So, we will be able to work out what its operation on any vector is because any vector can be expanded in this basis and then, it is a linear operator. So, we know everything about this operator right. So, these n squared coefficients have all the information necessary to completely define your operator right and for all practical purposes, we can just work with these coefficients; it is a square matrix right.

Although, A itself could be something you know a completely different beast altogether, it does not have to do with you know these kinds of matrices; but the point is that the algebraic properties of this operator are you know entirely contained in this representation right ok. So, once again I have to emphasize that representations are basis dependent.

So, in this basis, it looks like this this set of coefficients; but if you are considered some other basis, it would look different right and what happens when how these coefficients will transform when you change go from one basis to another is a topic for you know a future lecture; we will discuss that in some detail later on.

(Refer Slide Time: 07:30)



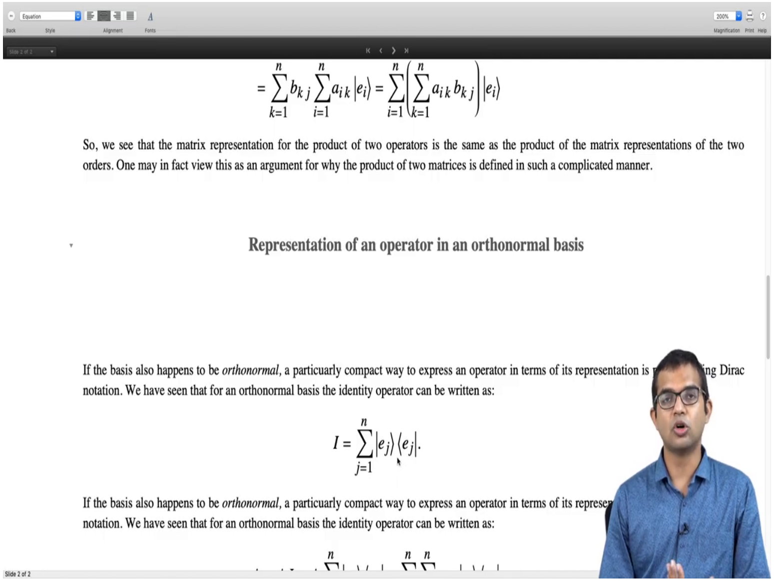
But now, let us look at what happens to the product of operators right. If we take an operator A and if we take another operator B and you know they have their own representations. a i j is one matrix which represents A in some basis and b i j is a representation for the matrix B. Let us say in the same basis, we are considering one basis; but looking at the representations of two different operators and suppose, we are interested in in the product of these two operators right.

So, the fact that a i j and b i j represent representations in this basis imply these two equations for A and B and then, we let us look at what happens if I were to multiply these two operators right. So, if I multiply two linear operators, I get another linear operator. Now, you know like I said an operator is what it does right.

So, if I can work out what this operator A B does to all the basis vectors, then I know everything about this product A B. So, what is A B acting on e j, it is the same as A you know follow with B acting on e j and close in brackets. So, I will first work out what B acts on e j, but which is already known to me.

What B does when it acts upon e j is just this, the summation over k b k j, I have used this index k, it is just to get summed over it does not matter whether I call it k or I call it i. But it is convenient to call it k here. So, then I expand this and I have summation over b k j. I bring in A right, it is a linear operator; so, it goes right through.

(Refer Slide Time: 09:07)



And then, A acting on e k itself is something that I already know right from these equations. So, then in place of A acting on e k, I write it as summation over i going from 1 to n a i k e i and then, I you know bring the summation to the left side and then, I exchange the order of these summations right. So, you know all these sums and exchanges of sums are completely legitimate because we are dealing with just finite dimensional matrices.

And so, there is no need to worry at all about convergence and all these kinds of issues, they do not arise here. So, we have summation over k a i k b k j. Now, but this object is a familiar object, it is really the multiplication of these two matrices as we already know a i k b k j will be the summation over k will be the ij'th element of the product of these two matrices A and B.

So, what we have managed to show you know you should think about it a little and convince yourself that we have managed to show that the product of two matrices, the representation of the product the representation of the product of two operators is equal to the is the product of the two matrix representations right.

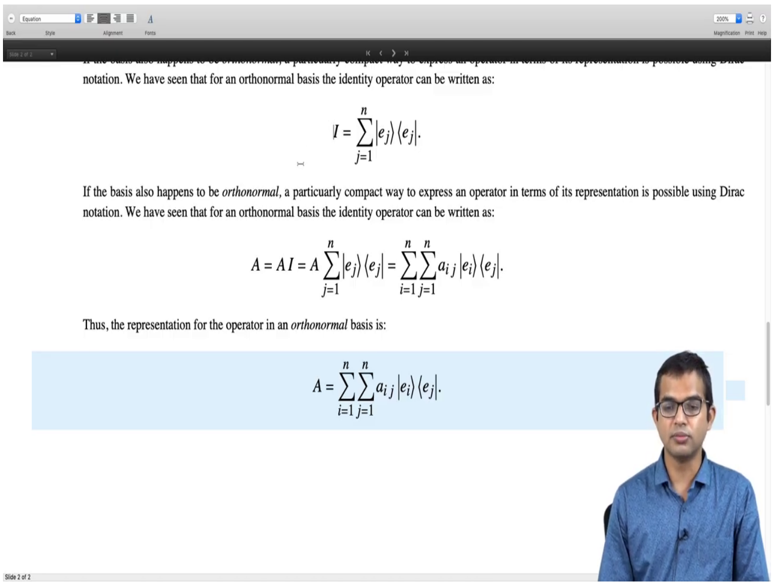
So, one might even argue that this is actually you know demanding this that you know in any basis, the representation of the product of operators must be equal to the product of the representations. You know one can argue should be the way one you know derives this somewhat complicated ordered operation for the product of matrices right.

So, you might demand this requirement that this represent the you know this product A B, the operator A B. So, therefore, this should be the definition for the product of matrices. But I mean we are already familiar with this right. So, this is if you wish an argument for why, how this complicated definition comes about ok.

So, representation of an operator in an orthonormal basis; so, if you have so far I have said, we have just some arbitrary basis and I have said the basis, the representation comes about by asking what an operator does to you know every element of your basis. Now, if I make it an orthonormal basis, then there is a particularly convenient representation right or form in which you can write down this you know operator and so that is the following.

So, if you have an orthonormal basis. So, we have seen that there is this identity operator. There is a nice compact way of writing the identity operator which is called the completeness relation right. You put together all these outer products of these basis vectors and then, that has got to be the identity operator.

(Refer Slide Time: 11:55)



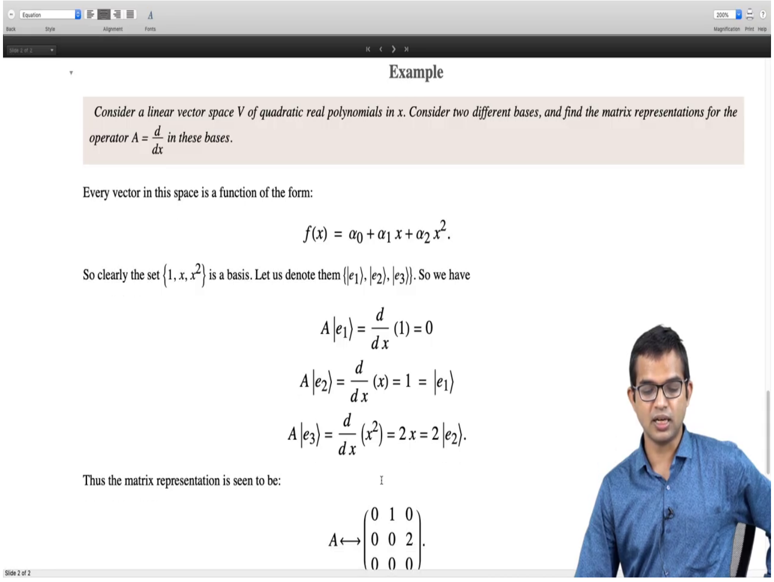
Now, this holds if e j are orthonormal. Now, we can multiply any operator with the identity operator for free. So, if I take a linear operator A, it must be the same as A times the identity operator. So, and for identity operator, I will substitute you know this expression; for I, if I plug it in here and then, I operate with A on e j.

I already know that A on e j, when it acts on e j, I get summation over i a i j e i alright. So, basically so that means, I have this nice expression for this operator in an orthonormal basis right. So, we have A is equal to summation over i summation over j a i j e i e j right. So, this is A.

So, another way of thinking about this is we can say that the ij'th, you know e I, if I bring in the Bra vector e i from the left side and the Ket vector e j from the right hand side. So, then e i e j'th matrix element of this operator in this orthonormal basis is just given by a a i j right. So, that is that the particularly convenient and compact expression and this trick of introducing identities operators for free is a recurring theme.

You will see this a lot in quantum mechanics, lots of manipulations become very very elegant, if you use this property and it works when you have an orthonormal basis ok. And you know you can introduce all kinds of orthonormal basis, you know which are which correspond to eigenvectors of different operators and so on right. So, these are tricks that you play and become comfortable with right, but keep this in mind.

(Refer Slide Time: 13:33)



Now, let me give you an example of you know I said that you can take an abstract operator and give it a matrix representation right. So, let me consider you know a linear vector space of quadratic real polynomials in x and let us look at two different bases and how the matrix representation for this operator A.

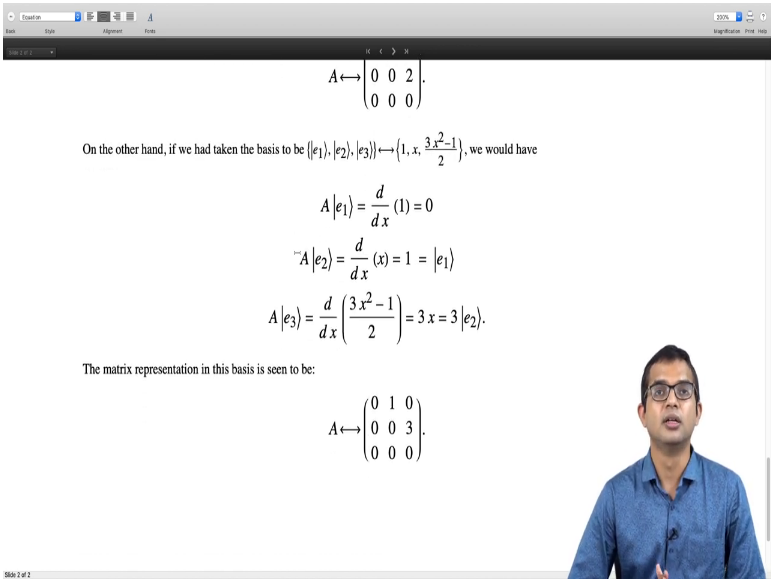
You know operator A also I am taking it to an abstract quantity. The derivative is an operator, it is a linear operator and let us see what this operator is, how this operator looks in two different bases. So, every vector in this space is a function of this form; f of x is equal to alpha naught plus alpha 1 x plus alpha 2 x square right. It is a quadratic polynomial.

So, you just need these three coefficients; alpha naught, alpha 1 and alpha 2. So, clearly this set 1, x and x squared is one basis, it is a natural basis to think of these kinds of functions right. So, we can denote them as e 1, e 2 and e 3. So, if we work out what the operator, the linear operator A does to each of these basis vectors, we know everything about this operator.

So, if I do A acting on e 1 which is the same as taking the derivative of just the constant 1 which is 0 and then, A acting on e 2 is the derivative of x which is 1 which is the same as the unit vector e 1 and when A acts on e 3, you get the derivative of x squared which is 2 x which is the same as 2 times e 2.

Now, we can put together all this information and form the matrix representation for this operator A in this basis and that will just turn out to be you know this matrix; 0 1 0, 0 0 2, 0 0 0. You know it is a very simple representation in this case.

(Refer Slide Time: 15:16)



So, on the other hand, if we had taken the basis to be e 1, e 2 and e 3 to be a different basis like I am free to select there are you know infinitely many different ways in which we could have come up with basis; 1 x and 3 x squared minus 1 by 2 is also an equally good basis right. So, you know when you look at orthogonal polynomials, you will encounter this kind of a set of polynomials.

But anyway, the point here is that you have another basis, you have another basis and if you operate with A on these basis vectors, you will get 0, e 1 and 3, e 2 this time. And so, therefore, the matrix representation in this basis seems to be you know 0 1 0; in place of 2, you get a 3 here right.

Not much has changed, but the point I am trying to make is that the same operator may look different, if you had take you can play with you know different kinds of basis and you know work out what this operator looks like in this basis and you can also play you know other games like take operators like d by dx or d squared by dx square.

And then, you try you can take you know products of these two vector of these two operators and see if you know explicitly computing the product of two different operators and working out the matrix representation agrees with the product of the matrix representations like we have seen or you can take the sum, you can do manipulations of operators.

You know first you do it with the operators themselves in an abstract manner and then, work out the same kind of operation we also hold for the representation itself and then, check that it is all consistent right. So, the main point of this discussion is to show that you know when you have linear vector spaces with finite dimensions, you can just think of all operators as linear operators as just matrices right and vectors as these column vectors.

So, in this sense, the theory of finite dimensional linear vector spaces is really the theory of matrices right. So, therefore, if we have a good solid understanding of matrices, we pretty much know everything about Finite Dimensional Linear Vector Spaces right. So, using you know this ah knowledge right.

So we will look at many properties of matrices in the lectures ahead and you know get a sort of you know we wind up our discussion of linear vector spaces. In the next level lecture, we are concentrating more mainly on matrices ok. That is all for this lecture.

Thank you.