Mathematical Methods 1
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Linear Algebra
Lecture – 31
Similarity and Unitary transformations

Ok. So, we have seen how operators operate on vectors and give you other vectors right. So, it is also useful to consider transformations which take operators and give you other operators, which is the subject of this lecture. And in particular, we will discuss something called a Similarity transformation of which Unitary transformation turns out to be a special kind of a similarity transformation that is the subject for this lecture ok.

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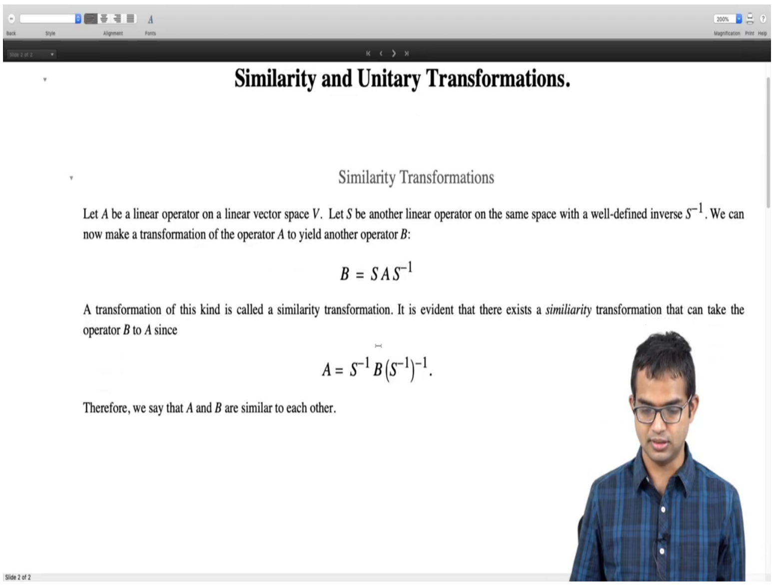

So, let A be a linear operator on some linear vector space V and let S be some other linear operator on the same space with a well-defined inverse right. So, S comes with an inverse S inverse and then, we can you know use S and S inverse together to create another operator right which is a transformation of the operator A which can be thought of as a transformation.

You will be using this rule right: you multiply S with A and multiply A with S inverse right. So, it is this. Product of these three operators is your new operator B right. So, transformation of this kind is called a similarity transformation right.

So, we are supposed to think of it as a transformation in which you know the input is A and the output is B right. So, it is evident that you know we could have gone to A from B as well and that is also possible using a similarity transformation right. Because you know you can see that A can be got as S inverse B times S inverse right.

So, S inverse is another linear operator right and it has exactly this form right. A is S inverse B S inverse inverse right. So, therefore, if B is obtained from A using a similarity transformation, then there is a way to go to A using another similarity transformation. So, we say that A and B are similar to each other ok.

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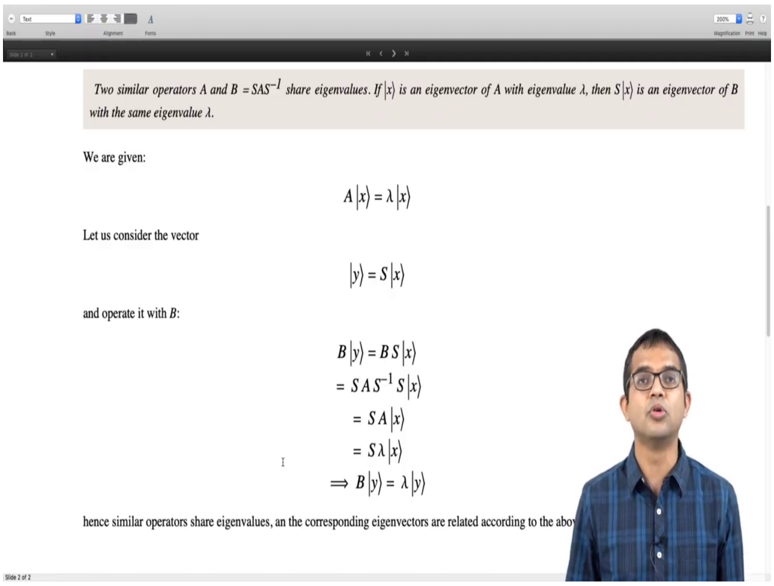

So, there is a crucial aspect of similarity transformations which we will prove now right. So, two similar operators A and B share Eigen values right. So, this has important consequences, we will return to similarity transformations at a later time, when we describe matrices and so on, but let us understand this crucial property right.

Similarity transformations leave the spectrum unchanged. So, if and in particular if x is an Eigenvector of A with eigenvalue lambda. So, then the Eigenvector corresponding to the same Eigenvalue for the operator B is going to be S times x right. It is very straightforward to see. So, let us look at the argument.

So, we are given that x is an eigenvector of the operator A. So, A times x with the eigenvalue lambda. So, A times x is equal to lambda times x. So, now let us consider this vector S, you know the operator S acting upon x will give you another vector. So, let us consider this vector y and so, let us see what happens when we operate with the operator B on this new vector y that we have just constructed.

So, if we operate with B on y, we have B times S times x because y is S x right. But B itself is S A S inverse and then, we also have S x. So, but then we can contract S inverse and S and so, we have S inverse S is the identity. So, S A S inverse S x is the same as S A x right.

So, what? But then, A x is the same as lambda x because x is an eigenvector of A right. So, this lambda will come out and then so, we have managed to show that is B acting on; B acting on y is yeah. So, I can pull out this factor lambda. So, then I am left with this just S x; but S x is the same as the vector y. So, B acting on y is the same as lambda acting on y. Therefore, in fact, the vector y is an Eigenvector of the operator B and with the same Eigenvalue lambda right.

So, hence, similarity transformation leaves the spectrum unchanged. And moreover, the eigenvectors of you know two operators which are related by a similarity transformation you know there is a this is well-defined and quick rule that is which we can use to get from eigenvectors of one operator to the eigenvectors of another operator provided the two operators are related by similarity transformation ok.

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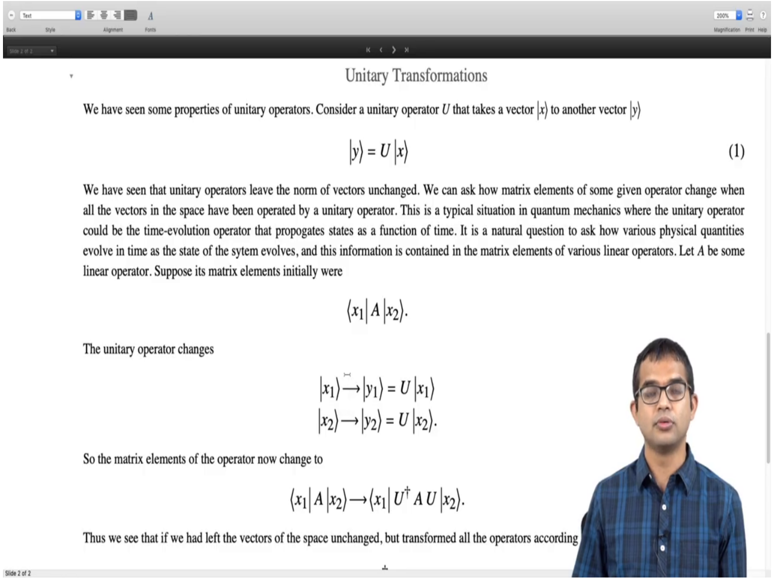

Now, let us look at a special kind of similarity transformation which goes by the name of unitary transformation. So, in order to get that, we will motivate it starting from unitary operators right. So, we have already defined what unitary operators are. Unitary operators are operators which are norm preserving operators right.

So, let us say you have some unitary operator U, it acts on some vector x and it gives you a vector y. We have seen that for unitary operators you know the norm of y is going to be the same as the norm of x right because of what unitary matrices are. This is like a defining property of a unitary operator.

Now, we can ask, suppose you take an entire space of vectors and you take all these vectors and then, you know, use the same unitary operator of all these vectors right. So, there were you know operators which lived in that space and we have seen that these operators are also defined in terms of the matrix elements right.

So, you can, you know, bring in a Bra vector and a Ket vector and compute matrix elements for some other arbitrary operator right. So, if you changed all these vectors, all these operators themselves are going to undergo a change right.

So, suppose you had some operator A right. So, some arbitrary operator A and you were looking at its matrix elements and they were given by x 1 A x 2, but because you have made this you know change, where you have taken all these initial vectors x and all these vectors x you know become U times x. So, then let us see what happens to the matrix elements of this operator, when all these x's have been changed to y's right.

So, let us so, if x 1 goes to y 1, x 2 goes to y 2. So, this matrix element x 1 A x 2 will go to x 1 U dagger A U x 2 right. So, so, that is the. So, the change of vectors causes a change of all these matrix elements. But if we stare at this equation a little then, we will see that there is an alternate way in which we could have thought about this.

We could have said this: suppose we do not do anything to the vectors, we just leave all the vectors as they are; but make a transformation to all operators right. So, and in particular to this operator A, if you had made a made a transformation U dagger A U right.

Then, you know basically, we are working with the same kind of change. The transformation could have been affected you know with operators in place of vectors right. So, this is something that happens in quantum mechanics, you might have seen this and now the ideas of the Heisenberg picture and the Schrodinger picture might come to mind right.

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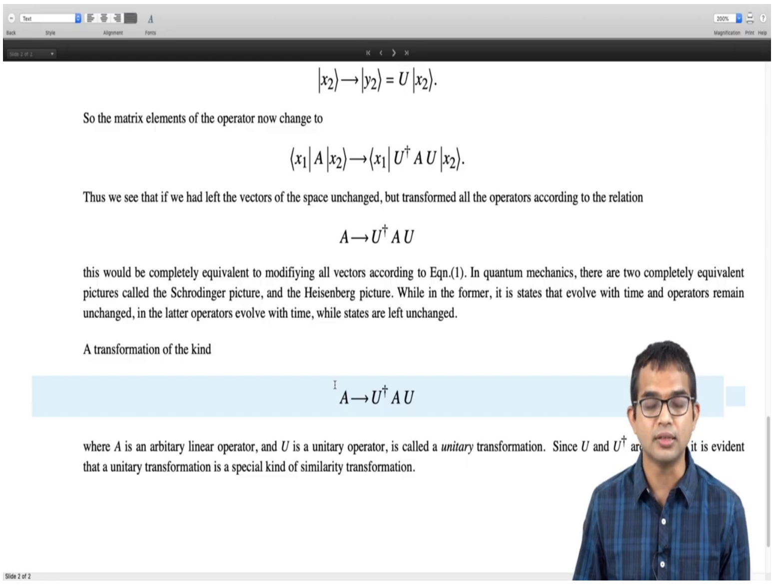

So, if we had made this kind of a transformation, where we take operators A and then, replace them by U dagger A U. This would be completely equivalent right and in quantum mechanics, all these kinds of unitary operators appear. For example, the context of time evolution operators right.

So, in the Schrodinger picture, we say that states evolve as a function of time and there is this unitary operator with which you act upon a state and you get the instantaneous state at a later time right. So, this is the Schrodinger picture and then of course, matrix elements will vary like here.

But on the other hand, you could also say that all the states remain as they are, but only it is the operators themselves which have a time dependence. This is called the Heisenberg picture in which the time evolution of the operator is given by a unitary transformation of this kind U dagger A U will tell you where the operator is right.

So, these are completely equivalent pictures and it is just useful to see that there are two different pictures in quantum mechanics right. So, but right now we are looking at this transformation itself right. So, from a linear algebra perspective, you know you can take operators and make them undergo transformations.

So, if you look at this type of a transformation which I called unitary transformation, we see that is this is in fact, a special kind of a similarity transformation and transformation of this kind, A going to U dagger A U, where U and yeah, U is a unitary operator is called unitary transformation.

And the unitary transformation, we can quickly convince ourselves is in fact a similarity transformation of a special kind because U and U dagger are inverses of each other right, which is the defining property of a unitary transformation is that is the adjoint of such a transformation is equal to the inverse right.

So, therefore, every unitary transformation is a similarity transformation and it is of a special kind right. So, that is all for this lecture. We will look at, we will come back to similarity transformations in the context of matrices and also, we look at how unitary transformations of this special kind have special properties associated with that. We will return to this a little bit later in the context of matrices.

Thank you.