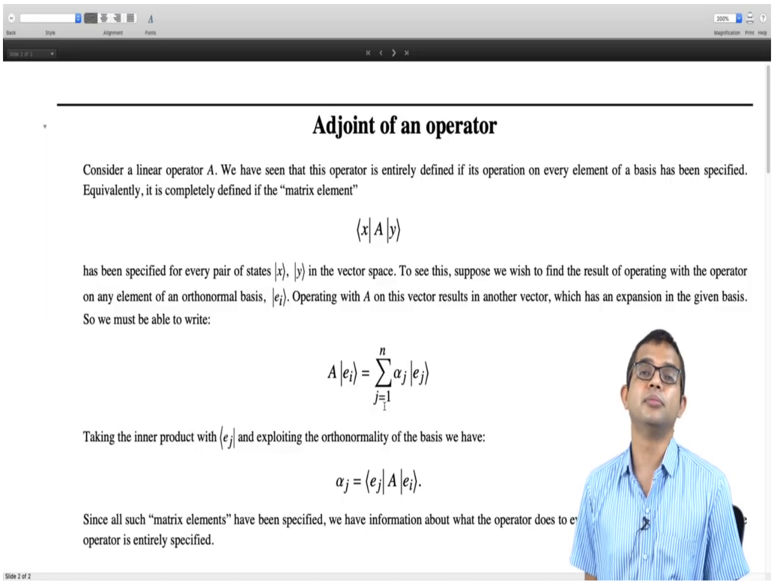
Mathematical Methods 1
Prof. Auditya Sharma
Department of Physics
Indian Institute of Science Education and Research, Bhopal

Linear Algebra
Lecture – 25
Adjoint of an Operator

So, in this lecture we will discuss the concept of the Adjoint of an Operator right. Sometimes it is also called the Hermitian conjugate of an operator. So, we will so, you might have encountered this in quantum mechanics. Some of these ideas that we will describe here will be useful for quantum mechanics right. So, this lecture is about adjoint of an operator ok.

(Refer Slide Time: 00:45)



So, consider a linear operator A. So, we have seen how this operator is completely specified if we are able to write down what its effect on every element of some basis is right. So, there is another way of specifying this operator which is in terms of what are called matrix elements right. If you are able to give you know the value of this bracket or this matrix element x A y for any pair of states x and y in the vector space if you are able to specify this, then that would entirely define your operator A right.

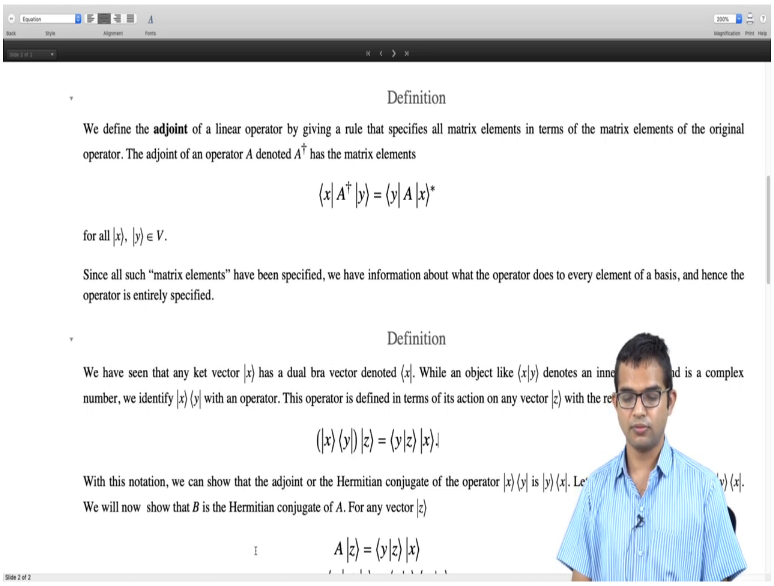
Given this result let us see how we are able to find out the effect of acting with this operator A on some vector e i. Let us consider some orthonormal basis e i. Now, if you operate with A on this vector it results in another vector. So, A times e i is another vector in your space. So, this vector itself has an expansion in terms of this orthonormal basis.

So, let us write it down you know abstractly as alpha j e j with the summation over j. Now, if you take the inner product e j and then exploit orthonormality you can immediately see that this alpha j the coefficients which are unknown are in fact, just these matrix elements e j a e i, but this has been specified for you right.

You know every matrix element possible for this operator A. Therefore, you know the effect of this operator on any of the basis vectors and therefore, you know everything about this operator right.

So, an alternate and complete way of specifying an operator is to just to give all the matrix elements right. So, we will use this way of specifying what is called an adjoint of an operator right. So, you are given an operator. So, which means that you know what the effect on all of the basis vectors of some basis is. So, we will specify every matrix element of the adjoint of an operator in terms of the matrix elements of the operator itself right. So, that is the way we will define the adjoint of an operator ok.

(Refer Slide Time: 03:20)



So, yeah it is given by this rule right. So, it takes a little bit of getting used to, but you might have seen the notion of conjugate transpose of a matrix. So, it is very similar to this idea and so, like I said earlier, operators are intimately connected to matrices right. So, we will discuss this as we go along, but for now we are still looking at these operators as abstract objects which are maps which take vectors you know to other vectors.

So, the adjoint of an operator also will take a vector and give you another vector right. But we will here define it in terms of matrix elements. So, it is simply given by the you know this relation. If you want to find the matrix element of x A dagger y, it is given by y A x star right. So, you must flip the ket and bra vectors x goes into the place of y and y goes into the place of x and you must also do this complex conjugation right.

If you do this you know that is the definition of a of the adjoint of this operator A right. So, since the operator A has been defined all these matrix elements of this operator are known and therefore, all the matrix elements of A dagger are known and therefore, A dagger itself is completely defined by this definition right.

So, let us look at you know the notion of you know taking a ket vector and a bra vector you know where the ket vector appears first and then the bra vector appears later on, right. So, we have seen that if you take the inner product of two vectors, we write it as you know the bra vector appears first and then the ket vector appears later on right.

So, every vector ket vector has a dual vector in the bra space and so, we have seen that an object like inner x y we know is a complex number right. Whereas, you know if you reverse the direction of (Refer Time: 05:29), if the ket appears first and then there is a bra that would be an operator right.

So, let us define what this operator is in terms of what it does to vectors right that is what an operator is. An operator is what it does to vectors right. So, let us define this you know sort of formally. So, you have an operator x y it acts upon z some other vector and it is simply defined as this inner product of y, z. So, you must think of you know this part becoming an inner product right.

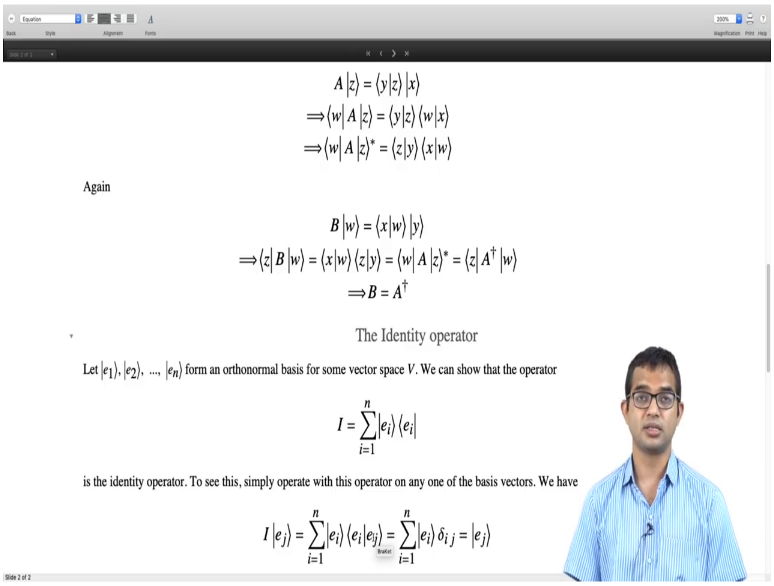
So, this is where this is in for in some sense the power of Dirac's notation right. So, you can take this you know bra vectors and then that will just merge with this ket vector z and you were left with this operator x right times this complex number which is the inner product of y and z right. So, yeah that is the definition of this operator x y right. With this notation you will see it is very powerful and it appears in quantum mechanics all the time right.

So, we should internalize it and you know the best way to do that is to work out lots of examples of this kind right and then once we get a familiarity it will be you know natural to think of operators in this notation right. So, with this notation we can show that the Hermitian conjugate of the operator x y is actually the operator y x right. If you just change you know the ket vector and the bra vectors you know if they exchange places then you get the Hermitian conjugate.

So, how do we see this? So, let us connect it back to the definition we just made for the adjoint of a linear operator right also or the Hermitian conjugate. So, let us call A is equal to x y this operator and B is the other operator y x right. Now, let us show that B is the Hermitian conjugate of A right. To do this we will work out some matrix elements.

So, for any vector z we have seen from the definition of this type of an operator A acting upon z is just the inner product y z times x right. So, this is by definition.

(Refer Slide Time: 07:54)



Now, we will multiply from the left hand side with some other arbitrary vector w right. We have a bra vector. Bra w acting on a A on z will be y z times w x right, both of these are complex numbers now right. So, the matrix element is a complex number in general, right. So, we started with a vector we multiplied this vector you know operated on it with a linear operator that gives another vector.

Now, you take an inner product from the left hand side with another vector w that will give you a complex number. Left hand side is a complex number, the right hand side is a complex number right. This is a check that you know one should do right. It is useful to do it explicitly when we are starting out when we are getting used to this type of notation.

Now, if we take the complex conjugate of this on both sides you know we see w A z complex conjugate of this is you know complex conjugate of y z star the inner product of y and z which is the same as the inner product of z a with y and this is also you know goes back to our definition of an inner product.

And the inner product is meaningful if this property holds, the inner product of two vectors is equal to the complex conjugate of the inner product of the same two vectors, but in the opposite direction right. So, likewise the inner product of w x star is going to be x the inner product of x and w. So, again let us look at what happens if you operate with B on this arbitrary vector w.

So, again using the definition we have we get x w and y and now if we operate from the left hand side if we take the inner product with z, then we have z B w is equal to this complex number x w times z y which is the same. We see it is just the product of these two complex numbers which is the same as w A z complex conjugate of this matrix element.

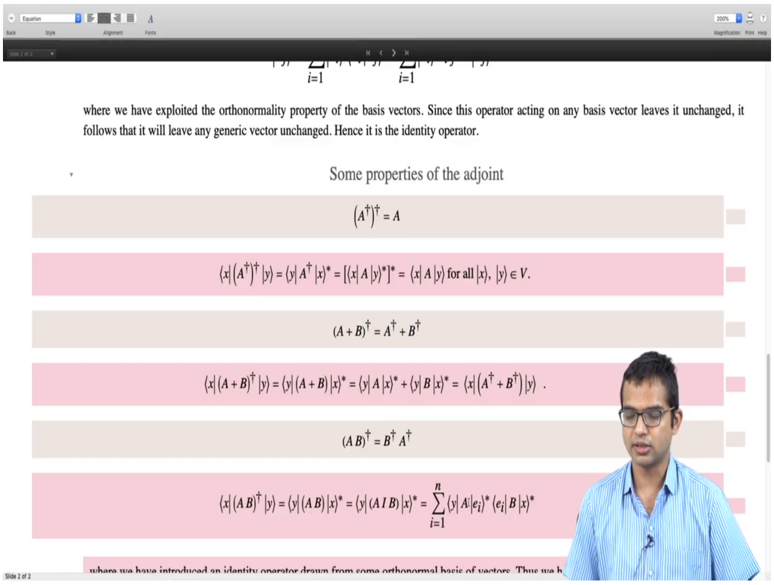
But what is this complex conjugate of this matrix element? We have seen that this is nothing but the matrix elements z w of the adjoint operator right. So, since this is true for any two vectors z and w. So, we have seen for any two vectors z and w, the matrix element of B is the same as the matrix element of A dagger, this is only possible if B is equal to a dagger right. So, that is how we managed to show that you know the Hermitian conjugate of the operator x y is operator y x.

So, let us look at how the identity operator can be represented in a very nice convenient form. Suppose you have an orthonormal basis for some vector space v, we can show that this operator is made up of e i e i. You take every such vector; you know take this the operator that we have defined corresponding to each of these vectors sum them all up that is going to give you the identity operator, right.

To see this you simply operate, you know use a definition you operate on any one of the basis vectors right. We have e i acting on e j is you know summation over e i, e i e i e j, but e i e j the inner product of e i e j is 0, unless i equal to j, then it will be one that is the same thing as putting a delta ij here which will give you e j.

So, what we have to managed to show is this operator i when it acts on any of these any of the basis vectors will just give you the basis vector itself and if this operator you know leaves every basis vector unchanged it means that it is going to leave every vector in the space itself unchanged which is the requirement for an identity operator.

(Refer Slide Time: 11:35)



So, indeed this is the identity operator right. We have used our new notation and then you know constructed some orthonormal basis and in terms of this basis this identity operator can be written in this very convenient form right. So, this is a form that will be used you know repeatedly when you are doing manipulations in quantum mechanics and so on; so, very convenient and useful identity to remember to keep in mind, ok.

Now, let us look at some properties of the adjoint. So, one is if you take the adjoint of the adjoint of a matrix you get back off an operator you get back the same operator itself right. So, this is something we can see invoking the definition right.

So, if you take any two you know vectors x and y, so we will work out the matrix element of this operator yeah A dagger is equal to by definition you know you have to flip y and x their positions go away and then in place of you know one of these dagger goes away you have A dagger x star of this.

Now, but what is this object? This again can be written as you know you have to flip x and y again and then you know the dagger is gone. There is another star, but you know the star and star complex conjugate for the complex conjugate of a omplex number is the original complex number itself. So, you will just get x A y and this is true for all x and y and that can only happen if this operator A dagger is the same as the operator A right.

So, this is the first property. The next property is A plus B the whole dagger is A dagger plus B dagger. Once again we can show this using the you know from first principles, using the properties of the adjoint of an operator. So, you take again any two arbitrary x and y and then you have A plus B the whole dagger in the middle, it must be equal to you know the vectors you know swapping places, y comes here first A plus B x complex conjugate of this.

But this is the same as you know these are like two complex numbers - I mean this also comes from the linearity. You can just write it as a sum of these two separately and then you can work this backwards and then you can rearrange terms again and convince yourself. Indeed change the position of x and y then you have A dagger plus B dagger.

So, the adjoint of the sum of two operators is equal to the sum of the two adjoint operators. And then finally, there is one more very interesting property which perhaps you have seen for matrices right. You have seen for transposes and for inverses also have a similar rule.

We should take the product of any two operators and then take the dagger of this operator which is the adjoint of the product of two operators is equal to you know the product of the adjoints, but in the reverse order right.

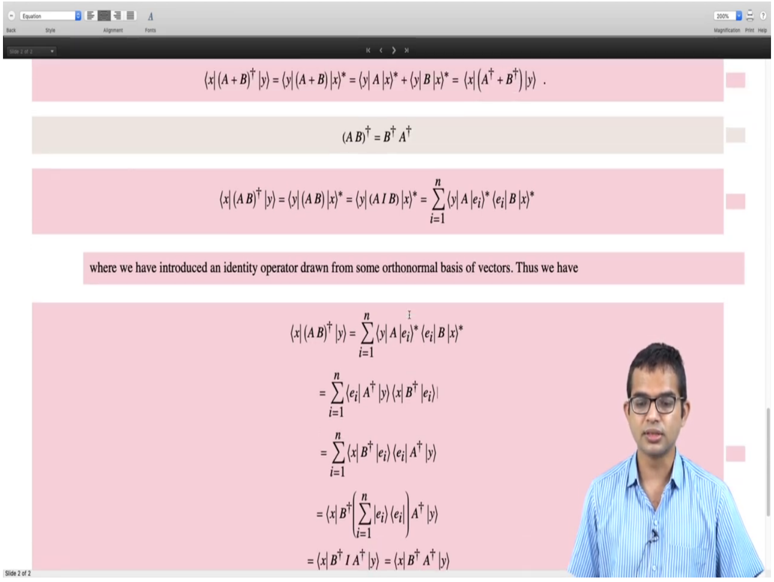
So, if you have many of them of course, this will extend. So, let us work out you know this rule for two operators, but the general you know n operator rule is something that you can work out on your own ok. Here we will exploit this identity operation and this also illustrates the power of this method right.

So, you are given x AB is the whole dagger y is equal to y AB x the whole star which is from the first property and then we introduce an identity that can be brought in for free right. Identity operators since it does nothing, so you just introduce this identity between A and B right.

So, these are standard techniques. So, it is worth observing this and you know this trick appears in quantum mechanics all the time right. So, A I B and then in place of I we introduce this you know what is called a complete set of states. So, you have some orthonormal basis for your overall space and so, you can write it as in terms of e i. So, you have y A e i star of this e i B x star right. So, and then you can you know I have just pulled out you know I have skipped a couple of steps here.

One is that you know I have brought in summation over I e i e i and then there is an overall complex conjugate. I can then pull out the summation and then since the complex conjugate of the product of two complex numbers is the same as the product of the complex conjugate that has also been used here right and then I can pull this out.

(Refer Slide Time: 16:36)



And now, this is what we managed to show. So, the inner product of the matrix element x A B the whole dagger y is equal to the summation of i y x e i star e i B x. So, I am just rewriting this. And now this can be rewritten as e i A dagger y. This is like the definition of the Hermitian conjugate of A and in place of e i B x the whole star, I will write it as x B dagger e i.

And then instead of writing this as you know the product of two complex numbers in one order is the same as the product of the same two complex numbers in the other order which is more suggestive. So, these two complex numbers exchange places right.

So, they look like there is a lot of activity here with lots of operators and everything, but really it is just the sum of the product of complex numbers right. So, I will be free to change the order of this because this is just multiplication of complex numbers.

And now you see that I have e i and e i which appear in this nice form. So, again I can bring. So, all the summation is contained only in this central region. So, I can bring the e i back in here and I can think of this as you know there is the inner product of x B dagger summation over i e i e i A dagger y. But what is this stuff inside the brackets? It is nothing but the identity operator right which we introduce right at the beginning.

So, now I can write this as x B dagger I A dagger y, but I does nothing. So, I might as well just remove it and then I have x B dagger A dagger y. So, since this is true for all vectors x and y, it must be the A B the whole dagger is equal to B dagger A dagger right.

So, this elaborate discussion illustrates the power of the identity operator and how it can be introduced you know strategically between operators and many results can be proved using this technique which will appear in many contexts in quantum mechanics ok. So, that is all for this lecture. More on this kind of identity operator decomposition will be used in the following lectures, that is all for this lecture.

Thank you.