Mathematical Methods 1
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Linear Algebra
Lecture – 24
Inverse of an operator

So, we defined what an operator is, we defined what a linear operator is. So, we will look at the notion of the Inverse of an operator in this lecture and some consequences based on the definition itself, ok.

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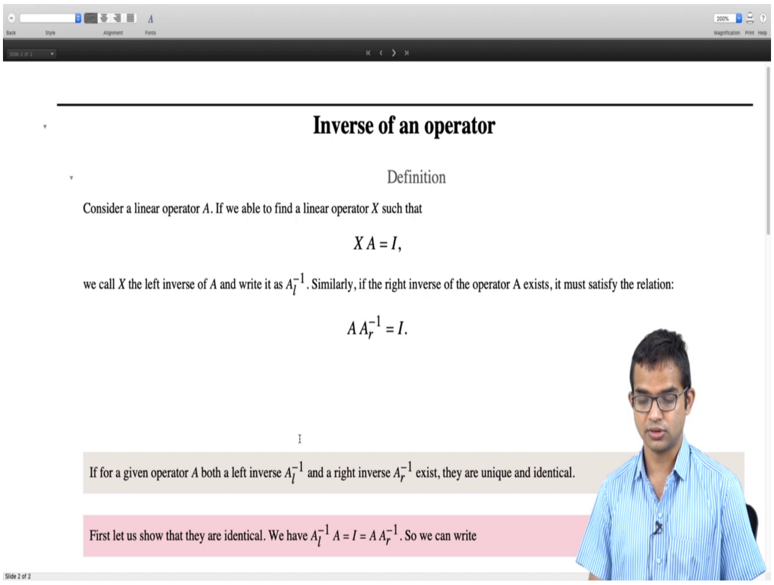

So, suppose you have some linear operator A and we are interested in some other linear operator X, such that X times A is equal to I, right. If you are able to multiply with this operator you know from the left side and such that the resulting operator is just the identity operator, then we call X the left inverse of A and we write it as A l inverse, right.

Similarly, we can define a right inverse for a linear operator A, if it exists then it must satisfy the relation A times A r inverse is equal to I, right. So, the existence and the relationship between the left inverse and right inverse is much more complicated if you are dealing with infinite dimensional spaces.

But in our discussions we are primarily going to stick to finite dimensional spaces where we are able to show a couple of very interesting and you know sort of results which can be obtained based on first principles. So, let us look at these results. One of them is that if for a given operator A, you know both a left inverse and a right inverse exist then they are unique and identical, right.

So, you know this just follows from the definition itself, right. So, you might have seen something like this in the context of matrices, right. So, later on we will see that you know operators are intimately connected to matrices, right. So, we will come to that in a few lectures from now. But let us look at you know operators in an abstract form for the moment, right.

So, you are given a left inverse and you are given a right inverse, so first we must show that they are identical, right. So, we are given that A l the left inverse exists and A r the right inverse exists. So, A l inverse times A must be equal to the identity, but this also is equal to A times A r inverse.

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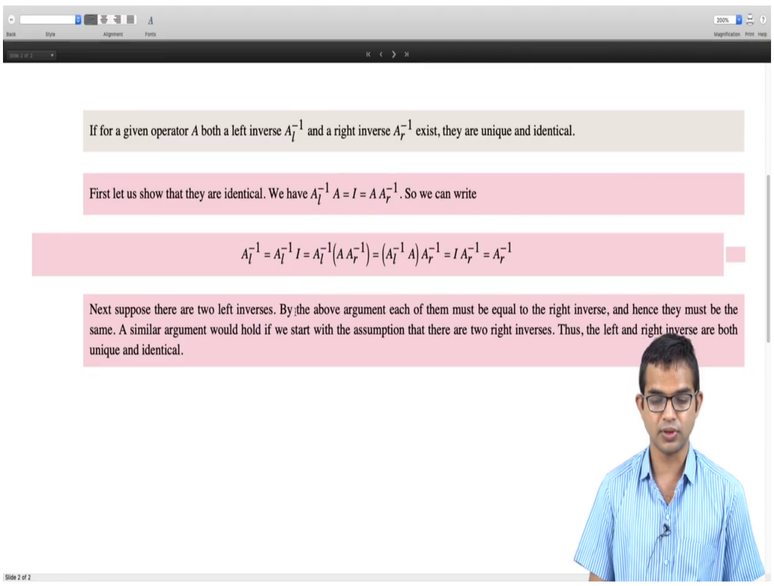

So, we are able to write A l inverse is equal to A l inverse times I, right, you can multiply with the identity operator for free, right. You know the property of the identity operator is that it will leave any other operator unchanged when it is multiplied by it. So, now, but in place of

I you can write A times A r inverse, right because we have you know we are given that there is an A r inverse.

Now, you rearrange these brackets, so then you have A l inverse A times A r inverse. So, but A l inverse A must be equal to the identity, so this is equal to I times A r inverse which is the same as A r inverse. So, we have managed to show that A l inverse is equal to A r inverse, right.

Now, so if there is a left inverse and if there is a right inverse they must be equal, right. So, this itself will in fact force both of them to be identical, to be unique, right. So, suppose there are two left inverses, now each of them must be equal to the right inverse, right.

If both of them are equal to the right inverse, then that means, they are both equal to each other as well, right. So, therefore, using this argument we can show that it is not possible to have more than one left inverse or more than one right inverse provided both the left inverse and the right inverse exist, right. So, that is one result, right.

I mean there are scenarios where you know one of these inverses does not exist and you have multiple of the other, right. So, that is the, you know unsaid fact. And it is true, you can find such examples when you have infinite dimensional spaces, but we are not going there at this point. So, let us say we are looking at finite dimensional spaces and we have a you know rigorous result which is that whenever you have a left inverse and a right inverse the two of them are the same.

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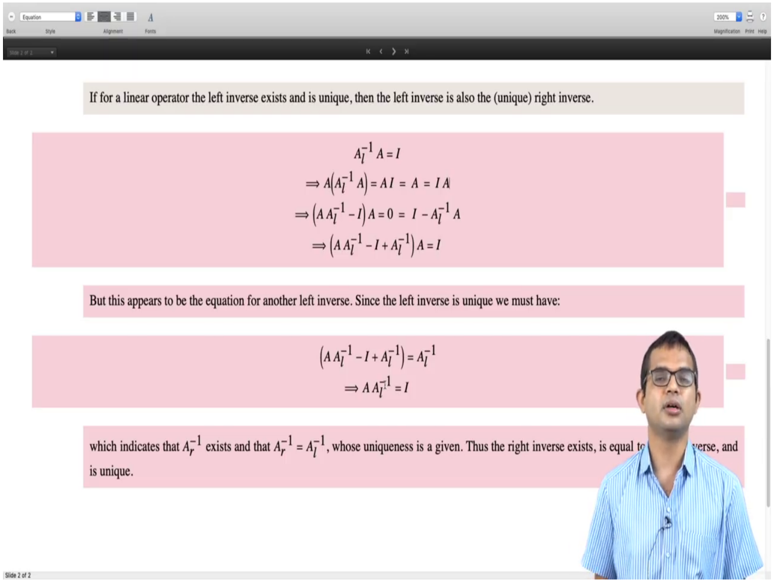

So, now, we have one more result which we can also show and the argument is very beautiful, so let us look at this. If for a linear operator the left inverse exist and is unique, right, I mean either you find a left inverse and a right inverse then you can show that they are unique and identical or if you are able to show that by some means that there the left inverse exist and it is unique then it forces that there is a right inverse and that also is unique, right. So, the argument is the following.

So, A l inverse A is equal to identity. Now, which implies, you can multiply throughout with A on the left hand side. So, you have A times A inverse A l inverse A is equal to A times I which is the same as A, but A can be written as I times A, right. So, now, the point is that we want to somehow get to A times A l inverse, right. So, we have A l inverse A is given, but we want to see what happens to A times A l inverse, right.

So, we will group you know the left hand side here and this here will bring this to the left hand side, so you have A times A l inverse minus I you know with an overall factor of A on the right hand side that must be equal to 0. But 0 is the same as I minus A l inverse A, right which comes from this first equation.

So, we have 0. So, this is 0 is equal to I minus A l inverse A. And then now look at you know this equation, you have this object is equal to this object. So, now we can bring this second term on to the left hand side, the leftmost you know side of this equation. So, then you have A times A l inverse minus I minus A l inverse times A is equal to I now, the right hand side is I because I have I am hanging on to just this I.

So, what have I managed to show? I started with the assumption that A l inverse A is equal to I, now I seem to have got some other operator times A is equal to I, right. But we know that A l inverse is unique, there is no second operator which there is no second operator X which when multiplied with A gives you I. So, this operator itself must be equal to A l inverse, right.

So, therefore, it forces A A l inverse minus I plus A l inverse is equal to A l inverse. So, if I cancel A l inverse on both sides, I get A times A l inverse is equal to identity. But what is this equation? This is the equation that A times some operator X is equal to I means this operator must be the right inverse.

So, but the right inverse here is A l inverse. So, we have the result that A r inverse exists and A r inverse is equal to A l inverse, right, whose uniqueness has already been given, A l if A l inverse is unique, so is A r inverse. So, if for a linear operator, the left inverse exists you know likewise you can argue from the right inverse as well.

If one of these inverses exists and if it is unique, then you are guaranteed that the other inverse also exists and it is unique. So, we might as well in these kinds of scenarios, we might just refer to this the inverse of an operator, right. So, you have a linear operator you just call its inverse as inverse.

So, you do not have to specify whether it is a left inverse or a right inverse as long as you know if one of them exists and if it is unique then the other also exists and that is also unique, right. So, these are all results which are perhaps obvious once you have seen it, but they all have been worked out starting from first principles and using a very nice beautiful chain of arguments, right. So, that is why we include this argument.

That is all for this lecture. We will continue with our discussion on linear operators in the next lecture.

Thank you.