Mathematical Methods 1
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Linear Algebra
Lecture – 23
Linear Operators

So, starting this lecture we will look at some properties of Linear Operators right. So, the word linear perhaps comes from these operators, that is why we tag along linear with linear vector space and linear algebra right. So, linearity is a key you know property that gets exploited in the many of the ideas that we are going to describe. So, let us formally describe what a linear operator is in this lecture ok.

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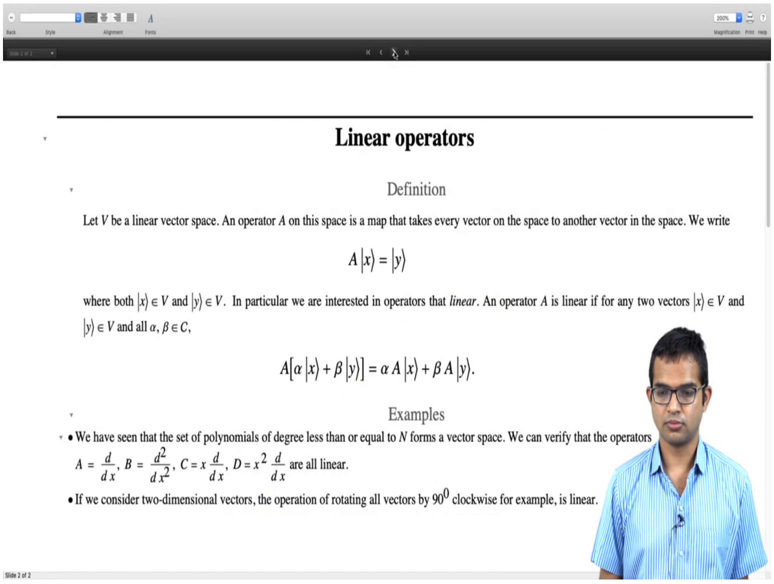

So, you have a linear vector space and an operator O right; so, is a map basically that takes every vector on the space to another vector in the space. So, we write you know A acting on the ket vector x gives you another ket vector y right, where both x is an element of V and y is an element of V for simplicity right. So, sometimes you know it is useful to define notions of domain and you know image and all this.

But for our purposes, let us just say that you are looking at you know operators which act on all states of your space and then, they give you another state within the same space right. So, in general an operator does not have to be linear right. But what makes it linear is the following property right. It is really the superposition principle right.

We have all seen the superposition principle in some form right which is that if you take you know some linear combination of vectors operate with this this map, this operator if it acts upon a linear combination of such vectors, it must give you the same kind of linear combination with you know the operator acting individually on the different vectors involved.

Suppose alpha, beta are some arbitrary complex numbers. A acting on alpha x plus beta y must give you alpha acting on alpha times A acting on x plus beta times A acting on y, that is what makes it a linear operator. So, let us look at a bunch of examples right.

So, we have seen that the set of polynomials of degree less than or equal to N forms the vector space right. So, what can you do with polynomials? You can take derivatives, you can come up with all kinds of functions which we will operate on, you can think of these functions as operators which act upon vectors which are these polynomials right.

So, if you take the derivative of this function, then you can verify using this property right. It is a linear operator and the second derivative is also a linear operator. You know x times d by d x x squared times d by d x, all of these are linear operators, as you should explicitly verify for yourself.

But on the other hand, something like if you if you are given a function if you are given a polynomial, if you take the square of this polynomial, if that is your you know the operation that you are going to do in it; then, it is not going to be a linear operation. It is just going to be a non-linear operation as you can verify right.

So, simply plug in two vectors and use your operation and check whether this condition holds. If not, then it is not linear right. You can also think of two-dimensional vectors and the operation of rotating all vectors by some angle right.

I have said 90 degrees clockwise; but any angle as long as you know all vectors are rotated by the same angle. So, then, you have a linear operation right. So, you can come up with operations on vectors which are non-linear in nature, just to contrast against this ok.

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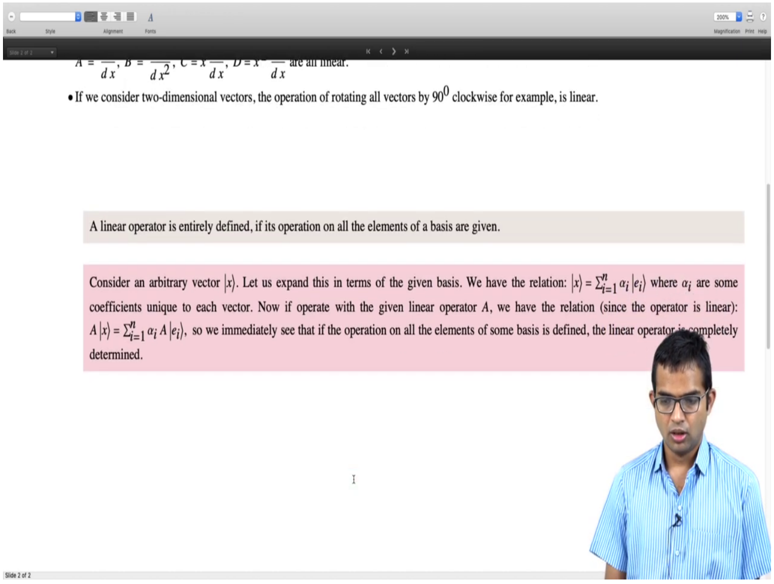

So, this is the notion of a linear operator. So, let us look at some immediate consequences of this definition. A linear operator is entirely defined if its operation on all the elements of a basis are given right.

So, we have seen that a basis you know consists of a bunch of elements and each of them contributes a non-trivial amount of information right. So, we have seen that it is convenient to have an orthonormal basis. But in general, a basis does not have to be orthonormal, but the total number of elements in a basis is fixed for a given vector space and it is equal to the dimension of the space.

So, consider an arbitrary vector x. So, let us expand this in terms of the given basis, which is always possible because it is a basis. So, we have the relation x is you know sum over alpha i; alpha i e i, where alpha have some coefficients which need to be determined right. Now, if we operate with the given linear operator A, we have the relation A acting on x because it is linear right. So, you can take this operator and act it individually on different vectors e i, the basis vectors.

So, immediately see that if you know what A ei is for every ei, then you are able to find out what an operator A will do to any vector x. So, therefore, this operator is entirely defined, if you were to know what its operation is on every element of some basis right. So, that is it goes back. So, where we have just used you know two key properties; one is that every state, every vector in your space can be expanded in terms of the basis and the second is the linearity of the operator ok.

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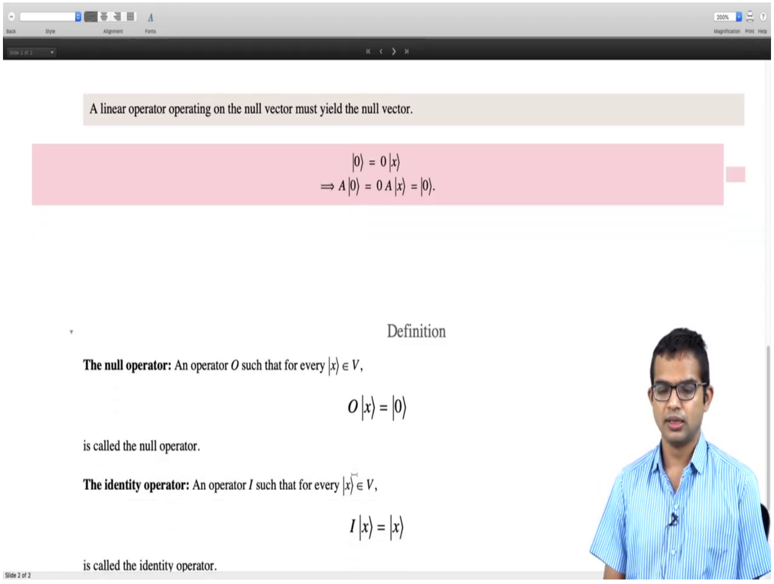

So another consequence; a linear operator acting on the null vector must yield the null vector right. This also comes from linearity and the basic property of another vector right. So, if you were to multiply any vector with the coefficient 0, you get the null vector.

And now, operate with your linear operator on both sides. So, you have A acting on 0 is the same as you know 0 times A x; but 0 times you know A x is some other vector. So, 0 times this vector also must be 0. So, A times the null vector is got to be the null vector. You cannot get some other vector if the linear operator acts on a null vector right.

So, we can also define two useful operators; one of them is called the null operator and the other one is called an identity operator right. So, the null operator is you know such that if an operator acting on any vector will just give you the null vector, then such an operator is called the null operator and the identity operator is an operator which when it acts upon every on any element of your vector space, it must just return the same element itself, then it is called the identity operator right.

So, this is a short and quick introduction to the notion of a linear operator and we will learn to work with this in the following lectures. Thank you, that is all for this lecture.

Thank you.