Mathematical Methods 1
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Linear Algebra
Lecture – 21
Gram-Schmidt orthogonalization

So, we have seen how in an n-dimensional vector space, if you consider an arbitrary you know set of n linearly independent vectors, these form a basis for you and a basis is a convenient you know set to work with because any vector in that space can be represented as a linear combination of vectors from this basis right.

Now, it turns out that although we can find a basis in many different ways, there are infinitely many ways; there are certain kinds of bases which are more convenient right; so these are called Ortho-normal basis right. And, 1 example is you know from two-dimensional vectors, we know that e x and e y right, we have perhaps without thinking about this, we have been using e x and e y to form the basis for all vectors in 2D right.

So, the vector unit vector along the x direction and the unit vector along the y direction; but in our recent discussion, we have seen that you know e x and e x plus y is an equally valid basis. We could have represented all our physics in terms of e x and e x plus e y or e x plus y and e x plus 3 e y for example or any two linearly independent vectors from the plane is that valid basis.

But you know the reason why e x and e y is a particularly convenient basis is because you know there is no component of e x along e y and there is no component of e y along e x. So, the idea here is that e x and e y are normal, are orthogonal to each other and each of them are themselves normal right.

So, that, so what it means is each of the vectors of your basis is bringing a unique piece of information for the set right. Together of course, they add up to the entire information of the overall space and there is no redundancy.

The fact that they are a basis implies that there is no collective redundancy; but if you make it an orthonormal basis right, if all the basis vectors are orthogonal, it means that no vector carries any information contained in another vector. That is what the orthogonal basis does right.

So, in this lecture I am going to describe a systematic method by which you can start with a basis of any kind, it is a basis and that can be made into an orthonormal basis from this right. So, this is called the Gram-Schmidt orthogonal orthogonalization process ok.

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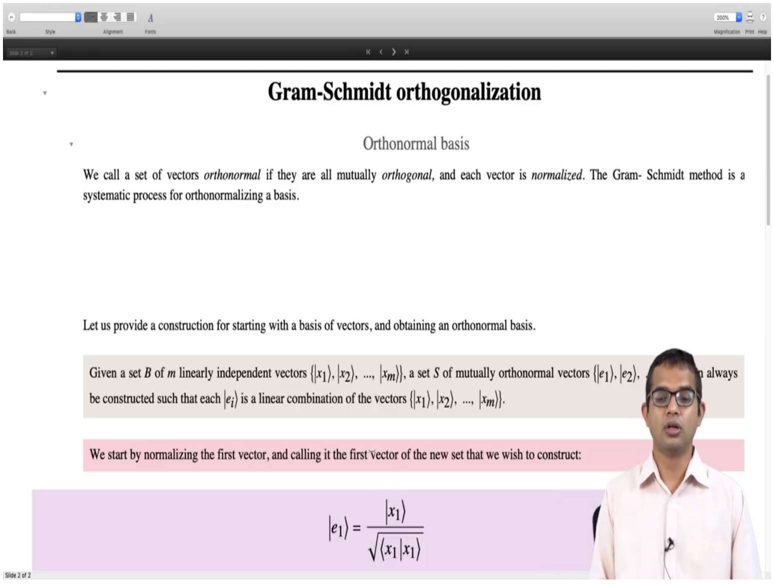

So, yeah; so, like I said, we call a set of vectors orthonormal, if they are all mutually orthogonal to each other right and each vector is normalized. So, making each vector normalized simply means you take the if you take the inner product of a vector with itself, you have to get 1 right. So, this is you know some added convenience even if they are not normalized.

So, it is just one more step to normalize it as well. So, we will show here how there is a systematic way to take any basis and make it and get an orthonormal basis ok. Given a set B of m, m linearly independent vectors; a set, a set of s a set s of mutually in orthonormal vectors e 1 to e m right; it can always be constructed that is the statement and so, how do we do this?

So, let us start by picking up any one vector right. Without loss of generality, we can just call it x 1. So, e 1 is equal to x 1 divided by its norm right so that e 1 has been normalized. So, the first step is to take any one vector and normalize it and that will be your first element in the basis that you are creating.

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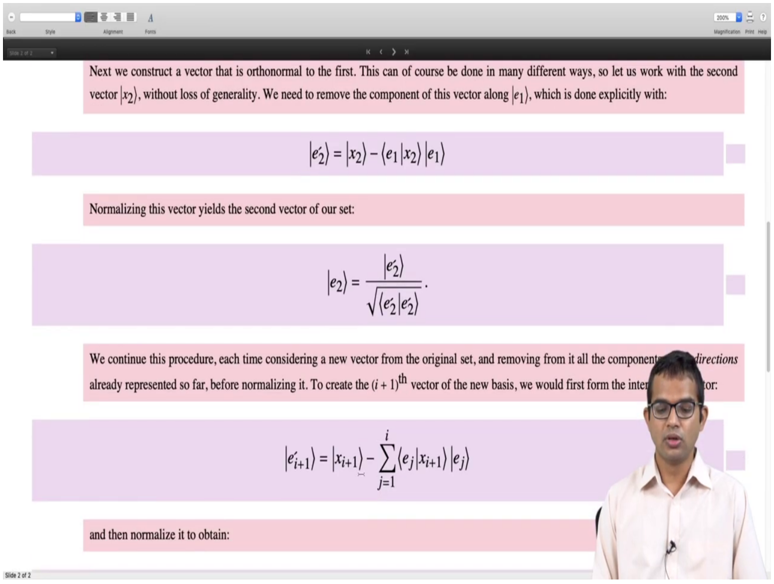

Now, the next thing to do is pick any other vectors. Now, you could have picked any one of the other n minus 1 vectors. There is no problem, there is a lot of choice in this right and in fact, you can create lots of different basis, now orthonormal basis using this method. So, I am just showing you that it is possible to find such an orthonormal basis in a systematic way.

Now, pick x 2 without loss of generality and then0, what you do is you peel off you know the component of x 2 which lies along e 1 right. So, x 2 is; so this is what we do. How do we find it? How do we peel off the component of x 2 along e 1? You just take the inner product of x 2 with e 1 you know and then, attach this vector e 1 and then, you remove that component from x 2.

So, this is an intermediate vector which we call e e 2 prime right and to get e 2, we have to just normalize this vector right. We are interested in creating an orthonormal basis. So, we have to normalize this e 2 prime. So, then we have e 2 prime divided by square root of the inner product of e 2 prime with e 2 prime that is the second vector e 2 right. By construction, you can check that this vector e 2 is going to be orthonormal, orthogonal to e 1 and it is also normalized by construction right.

So, the next step is similar right. Now, you have to peel off, you take the third vector x 3 without loss of generality and will peel off, you know the component of x 3 along x 2 and the component of x 3 along x1 and normalize it and so on.

And then, you keep on doing this for the ith vector the i plus 1th vector is going to be obtained by peeling off from the i plus 1th vector you know components along every vector which has already been put into your box of orthonormal vectors that you creating right. So, e i plus 1 prime is going to be x i plus 1 minus summation over j e j x i plus 1 the inner product times the vector ej right.

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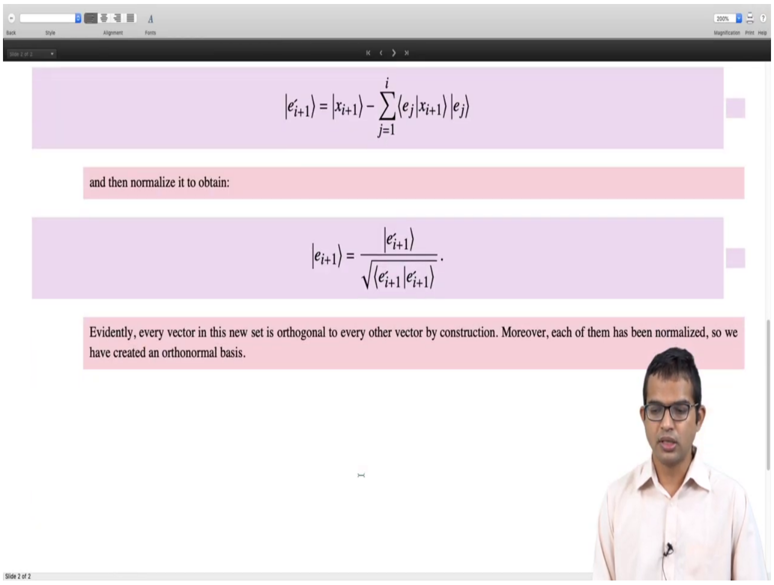

If you do this and then, you have to normalize it. You have to do each e i plus 1 prime divided by square root of the e i inner product of e i plus 1 prime with e i plus 1 prime and then, so evidently, you can see that by construction every vector in this; in this set is orthogonal to every other vector. And also, every vector has been explicitly normalized. So, you have created an orthonormal basis.

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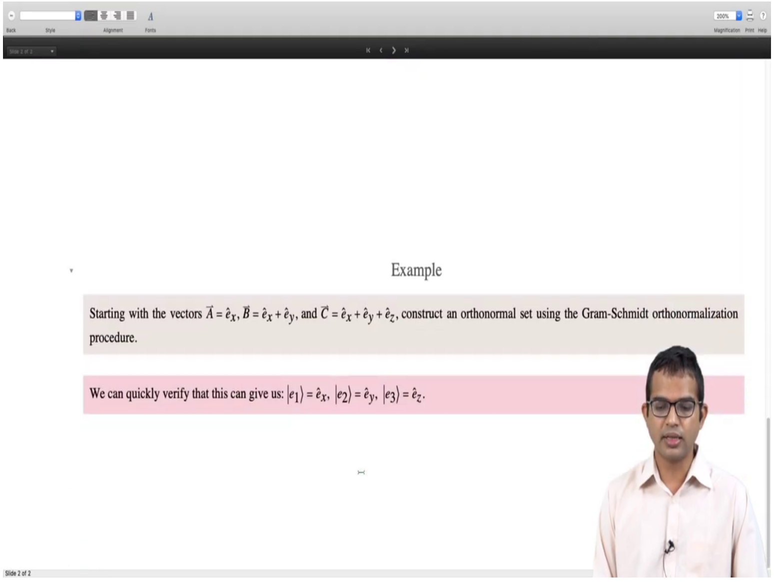

So, that is how this method works. To convince yourself that this works out, I have given you an example which is very very simple and you might not need an elaborate method for this. But let us say that you start with the vectors e x, e x plus y and e x plus e y plus e z right. I am thinking of a three-dimensional space, you know the usual kind of vectors. e x is given; e x plus y e y is given and e x plus e y plus e z is given right.

You can, if you take the first vector to be A itself right. So, it has already been normalized. So, you have e 1 is equal to e x and then, when you go to the second step e 2. To create e 2, you are going to peel off you know the component of B along e x which is in this case will turn out to be just e y and then, you go to the third vector C. You peel off the vector along A and the vector along B. So, then you will get e z.

So, e x, e y and e z is not just a basis; but it is an orthonormal basis. So, I have given you a very trivial example. But even with this trivial example, let us say you start not with a vector A, you I said even within the Gram-Schmidt orthonormalization processes itself, you have a lot of choice.

Suppose, you start with a vector C, you know make C the vector e 1 right. You have to do e 1 C divided by mod of C 1 mod of C, that will be e 1 and then, maybe you can take B as it using B you have to find e 2 and find A, use A to find e 3 right. That would be left as a homework exercise. You could also try starting with B, you know try out various different combinations and see how you can generate different kinds of orthonormal basis right; that is all for this lecture.

Thank you.