Mathematical Methods 1
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Linear Algebra
Lecture - 14
Determinants and their properties

So, in this lecture, we are going to discuss Determinants and some of their properties right. So, I am going to assume that all of us have seen determinants at some level, and we have also played with them and used their properties and so on. But, so in that sense it is going to be a kind of recall of what we have already seen.

But so the idea here is to collect together a bunch of results; which are all related and which can be written down very neatly if we use determinants right, so that is why I am going to bring in determinants here.

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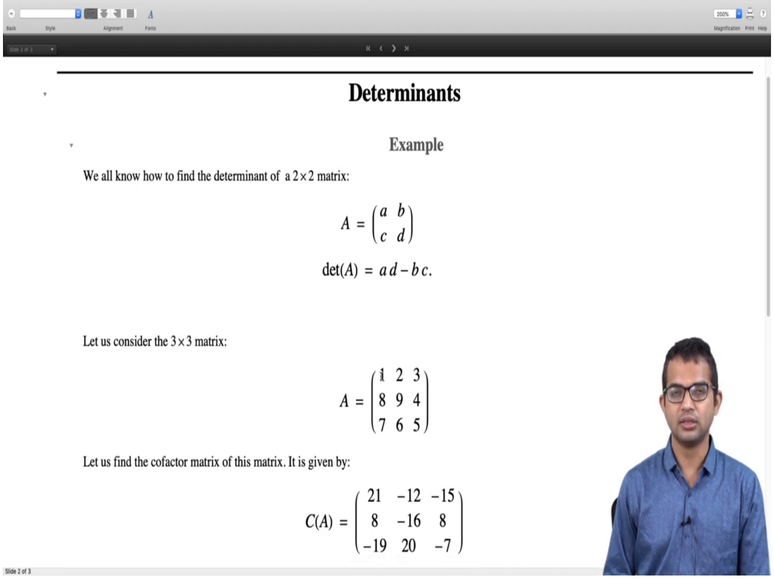

So, let us just you know start with 2 by 2 matrices. And so first describe how to compute a determinant you know in this very standard in an algorithmic way right. So, there are different ways in which one can introduce determinants and you know come up with the definition of determinant and so on. But, so we will not take a formal approach here. So, let us show how to do it for a 2 by 2 matrix and 3 by 3 matrix and so on right.

So, 2 by 2 matrix is you know a b c d. So, we know that the determinant of such a matrix is simply given by a times d minus b times c right. So, if you look at a 3 by 3 matrix the determinant is you know can be computed in terms of determinants of other 2 by 2 matrices. And so let me also point out that the determinant is defined only for a square matrix right.

So, a matrix in general need not be square as we have seen examples of non-square matrices, but the notion of a determinant is available only if the; you know a matrix is square.

So, if you have a 3 by 3 matrix, one should start with finding out what is called the cofactor matrix right. So, every element of this matrix is if it is replaced by its cofactor, then you get the cofactor matrix right. So, how do you compute the cofactor of an element? So, let us look at the first element here.

So, the first row, first column, so, that is 1. So, the way to find its cofactor is to ignore the row and the column that this particular element represents. So, you forget about you know this, this row, and you also have to forget about this column. So, imagine that this does not exist. So, then you are left with just this 2 by 2 matrix right, you have to find its determinant right.

So, what is the determinant of the sets? 9 into 5 minus 4 into 6, that is just 21 right. So, the cofactor of 1 is 21 right. So, you have to be a bit more careful, right. There is also the sign involved when you are computing the cofactor right. So, what I just told you is what is called a minor, and then you have to associate you have to tag along a factor of minus 1 to the row plus column right.

So, let us look at the second entry here 2 right. What is its minor? See you have to forget about this row and this column and so then you are left with 8 times by 40, 40 minus 28, 40 minus 28 is 12 right. So, but I have written minus 12 here. The reason it is minus 12 here is because the this is the first row and second column. So, 1 plus 2 is 3. So, you have to do minus 1 to the power 3 right.

So, the first you know element that we considered belongs to the first row and first column, so 1 plus 1 is 2. So, minus 1 square gives you a positive sign right. So, you get a minus 12 and minus 15 and so on. So, I will allow you to verify that I have done it correctly. So, you can find the cofactor matrix of any square matrix right by replacing every element by its cofactor right. It becomes tedious if you have very large matrices, but this is a prescription which will generalize even to higher dimensions right.

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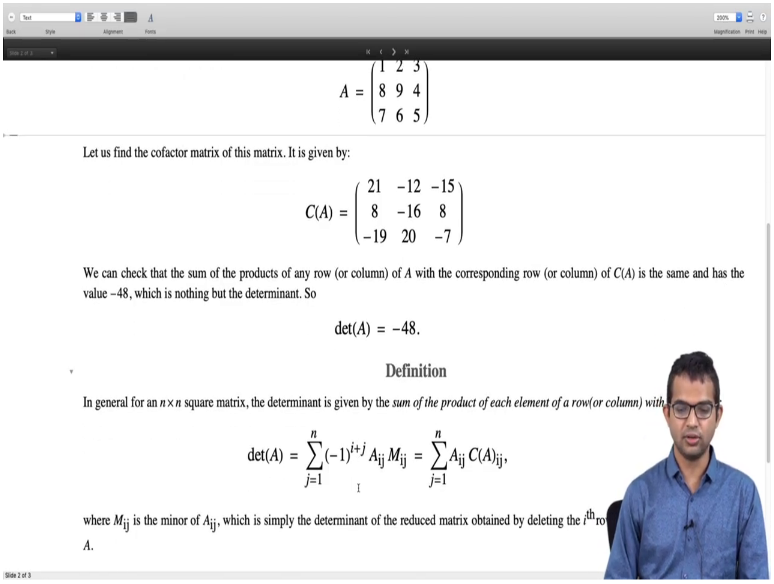

Once you have this cofactor matrix, what you can observe is if you multiply any row of this original matrix, let us look at the first row of this original matrix and the first row of this cofactor matrix. So, if you take you know 1 times 21 plus 2 times minus 12 plus 3 times minus 15, you are going to get minus 48 in this case. And likewise you can check if you do 8 times 8 plus 9 times minus 16 plus 4 times 8 that also is going to give you minus 48.

And so in fact, any row with any row corresponding row of the cofactor matrix if you know in some sense you are taking like a dot product of this right, so although you should not be calling it is just you have to multiply the corresponding elements and add them up you can do it with rows or you can do with columns, you will get the same number minus 48.

So, what is this magical number? Well, this turns out to be the determinant right. So, there are other ways of thinking about a determinant, but let us not go into this. So, this is one, one way to compute a determinant. And usually it is not you know so one does not usually compute determinants of large n by n matrices by hand.

Unless there are some special properties involved which you know special symmetry is involved in some matrices, then you can also you know write down the answer for an n by n matrix. And then typically you do not use this kind of a brute force technique involving cofactor matrix and so on right, but the point is that it is useful to know that there is this brute force approach.

So, once you have a definition for the determinant of a 2 by 2 matrix, you can define you know that you can define the determinant of a 3 by 3 matrix from which in turn you can define the determinant of a 4 by 4 matrix and so on right ok. So, this is the determinant. So, I am formally defining it here.

So, in general for an n by n square matrix, the determinant is given by the sum of the product of each element of a row or column. It does not matter which row you choose, it does not matter which column you choose as long as you take the same column and same row; you know for the matrix and its cofactor matrix right. So, your determinant of A is a summation over j minus i to the i plus j.

So, this is where this minor becomes a cofactor when you tag this factor along with it. So, M i j is the minor, A i j times M i j and then so which is equivalent written in terms of this cofactor of A i j right. So, this is exactly what we have demonstrated earlier already.

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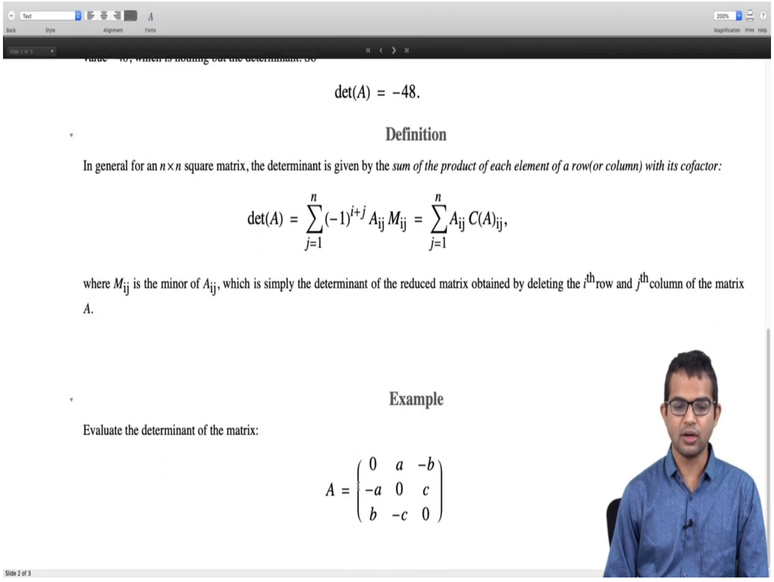

So, let us work out the determinant of this matrix ok.

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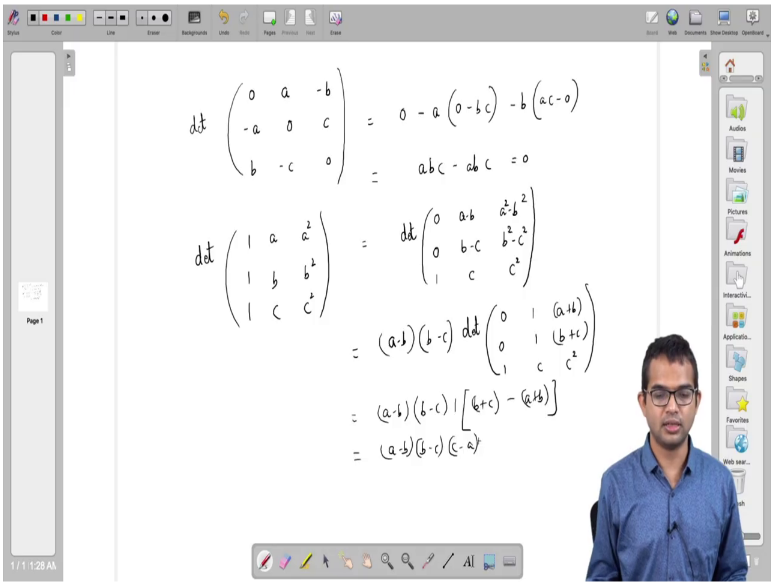

So, let me copy this down. I have 0 a minus b minus a 0 c minus a 0 c, then I have b minus c 0, and we can do it a brute force way. So, whenever you have 0s like this it makes life even easier right. So, let me go along this first row, so I have this is equal to 0 minus a times a into

0 will give me just 0 minus b c, then I have minus b times a c minus 0. So, what do I have? So, this is equal to a b c minus a b c equal to 0 right.

So, I have worked this out just using brute force method. But there may be other ways of doing this using some of some properties of determinants right. So, let us look at some properties of determinants, and maybe later on you can try to find if there is a quick and direct way to get to this answer that could be your homework. But let us look at some properties of determinants. So, this too is in the nature of recall.

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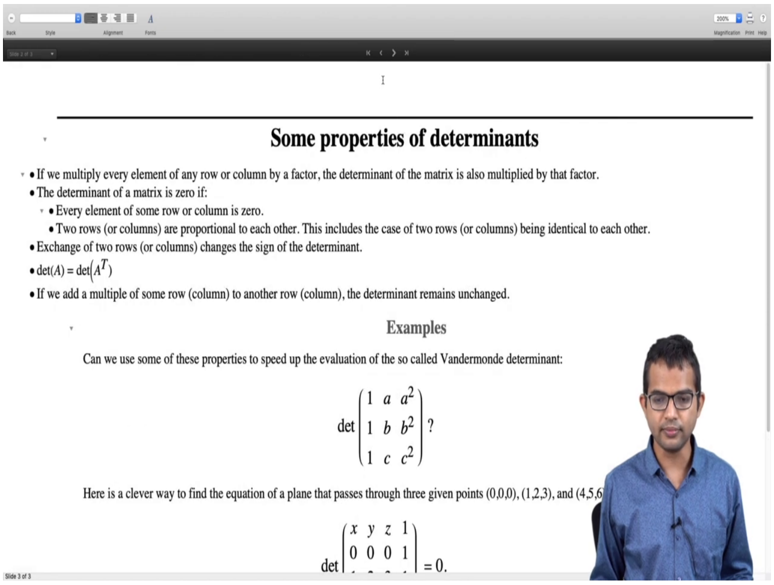

So, if you multiply every element of any row or any column by factor right. So, the determinant of the matrix is also multiplied by that factor right. So, this is something that you must be familiar with right. It does not matter whether you do it for a row or for a column, and it does not matter for which row or which column you do, the same factor you know can be pulled out or brought in right from the determinant.

So, the determinant of a matrix is 0 if every element of some row or column is 0. So, in fact, this is actually a consequence of the first rule right. So, if every element of some row or column is 0, then you can pull out 0 right. And therefore, the determinant of this matrix is equal to 0 times the determinant of this matrix so which is just 0 right.

So, two rows or columns are proportional to each other, so if two rows or two columns are proportional to each other, then this includes the case of two rows being identical to each other. Whether they are identical or proportional each of them will go to 0 - this is kind of obvious because you can pull out that factor and then there are two columns which are identical. So, and that gives you a 0 for the determinant right. This also follows from the fact that you can add or add a multiple of some row to another column, and the determinant remains unchanged which is a property which is coming up later on.

So, if you exchange two rows, it changes the sign of the determinant. If you exchange two rows or if you exchange two columns, then the determinant of your resulting matrix is going to acquire a minus sign right; if you do it multiple times, then it is going to give you know minus one to the power of so many times it should have done this exchange operation.

So, the determinant of A is the same as the determinant of A transpose right. If you add a multiple of some row or some column to another row or column respectively, the determinant remains unchanged right

So, you know we saw when we had a when we were doing this row operations right, so you could take a row and add it to the multiple of another row and so basically the information content in the original set of equations and the later set of equations were unchanged right. Now, we have specialized to the case where you have an n by n matrix right, and then so the value of the determinant of this n by n matrix is unchanged if you do these kinds of operations.

So, let us look at some examples of how you know these properties play out. So, let me, so I will allow you to see if you can find you know this determinant using some of these properties, but let us look at this so-called Vandermonde determinant. So, you have the determinant of this matrix 1, a is 1 let me write it down 1 1 1 a b c a squared b squared c squared right.

So, we can go ahead and subtract the first row from the second row. So, this is equal to the determinant of if I subtract the second row from the first, so I get 0, then I get a minus b, then

I get a squared minus b squared. And likewise I can subtract the third row from the second row. So, then again I get 0, then b minus c, then I have b squared minus c squared.

So, let me leave this row as it is 1 c c squared, so then I have. So, this is equal to. So, then I see that in the first row there is this common factor a minus b that I can pull out and in the second row I have this common factor b minus c which I can plug. So, I have a minus b times b minus c times determinant of 0 1 a plus b, and then I have 0 1 b plus c, and then I have 1 c and c squared.

So, now of course, I can just evaluate this determinant going along this first column, so then I just have 1 times this determinant that I have to evaluate. So, I get a minus b times b minus c times 1 into b plus c minus a plus b which is the same as c minus a right. So, I have a minus b into b minus c, it is a cyclic product into c minus a.

So, this is called Vandermonde determinant right. So, I have used some properties of determinants and I have evaluated you might be able to get to the same result using some other properties, some other technique which I will encourage you to try and find.

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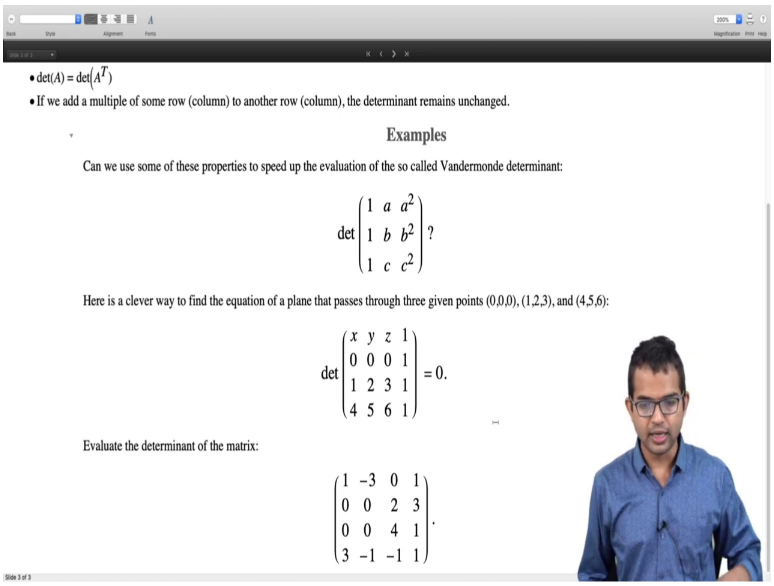

So, let us look at one more example which is a clever use of the properties of determinants. And then there is one more example which I will just allow it to be homework right. So, there is a clever way to find the equation of a plane that passes through three given points. So, you are given some three points x 1, I could have just written it as x 1, y 1, z 1, x 2, y 2, z 2, and x 3, y 3 z 3 for you know for concreteness I have put values in 0 0 0 1 2 3 4 5 6 right.

So, now the point is that you know a plane is completely the plane passing through these two points is completely different because you have given three points. Now, we know from coordinate geometry that the equation of a plane will be of the form some constant times x plus some other constant times y plus some other constant times z plus some other constant equal to 0 right.

So, I am claiming that this object - this determinant is equal to 0 is the equation of the plane right. How do we see this right? So, let us look at expanding this along the first row.

So, I have x times determinant of this 3 by 3 matrix plus y times the determinant of another 3 by 3 matrix which is some constant plus z times the determinant of some other matrix which is a constant plus another determinant which is a; which is a constant is equal to 0 which is exactly in the form of the equation of a plane right.

So, first of all, it is clear that this is the equation of some plane. Now, how do I show that this is the equation of the plane that we are interested in? So, I just have to show that if I plug in place of x y z, if I put in 0 0 0, then indeed this equation must hold that becomes evident.

Because I have 0 0 0 1 if I put it here then I see that the first row and the second row are identical. If they match, then we have this property of a determinant that whenever any two rows or any two columns are the same, then the determinant of such a matrix is 0.

Likewise, I see that if I plug in 1, 2, 3 here the values corresponding to the second point, then once again I will see that the first row and the third row are identical. So, this determinant is going to be 0. And if I plug in the third point 4, 5, 6, once again I find that the first row and the fourth row are identical, therefore, the determinant is 0 right.

So, all three the all three points definitely pass through this plane, that we have found. And it is the equation of a plane which passes from these three points. So, it must be the equation of the plane that we are interested in, so that is the argument. It is a clever application of the properties of determinants.

So, finally, I have one more example which I will allow you to work out on your own as homework. So, here the idea is you know whenever you have these 0s, it is a 4 by 4 matrix. You might think that it is a difficult job to evaluate such a determinant, so that you just go along this first column.

Since there are all these 0s, you will have to evaluate you know the 3 by 3 matrix which is correct you know which gives you the cofactor corresponding to 1, and then you have to evaluate the other 3 by 3 matrix, the determinant of this 3 by 3 matrix and you will be done. You have to take care of the signs properly, and then you will find that in fact, these 3 by 3 matrices themselves have 0s in 1. So, it becomes quite straightforward.

So, you can also try to do it using some of the properties of these determinants, but for this particular problem it is straightforward enough to just directly use the brute force definition right. So, this was a short lecture, recalling some of the properties of determinants and building the scene for our connection to linear algebra and (Refer Time: 18:13) linear equations which we will describe in the next lecture.

Thank you.