Functional Analysis
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Lecture 2.3
Hahn-Banach Theorems

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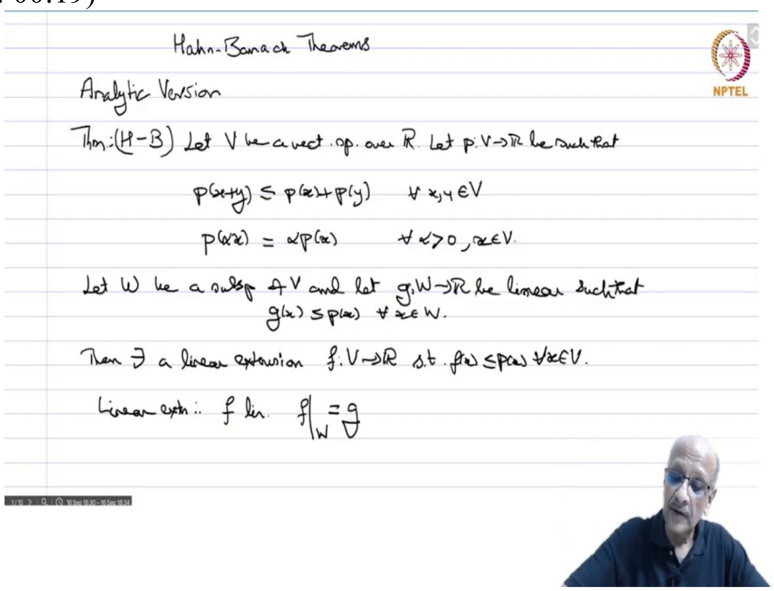

Hahn Banach Theorems.

There are several theorems which go under this name. Basically the same thing and there are essentially two kinds of Hahn-Banach theorems. First one is called the analytic version, which deals with the extension of continuous linear functional defined on a subspace to the whole space. The second one known as the geometry question deals with how you can separate convex sets in the vector space by means of hyperplanes.

Analytic Version. We are going to first prove a fairly general result and then we will deduce the extension theorem as a consequence.

Theorem. Let V be a vector space over R (now I am specifically saying that this is a vector space over R) and let $P: V \mapsto R$ be such that $P(x+y) \leq P(x)+P(y), \forall x, y \in V$ and $P(\alpha x) = \alpha P(x), \forall \alpha > 0, x \in V$. Let W be a subspace of V and let $g: W \mapsto R$ be linear such that $g(x) \leq p(x), \forall x \in V$. Then there exists a linear extension, $f: V \mapsto R$ such that $f(x) \leq P(x), \forall x \in V$.

So, what do we mean by linear extension? This means, f is linear and f restricted to W is the same as g. We are going to prove this using Zorn's lemma.

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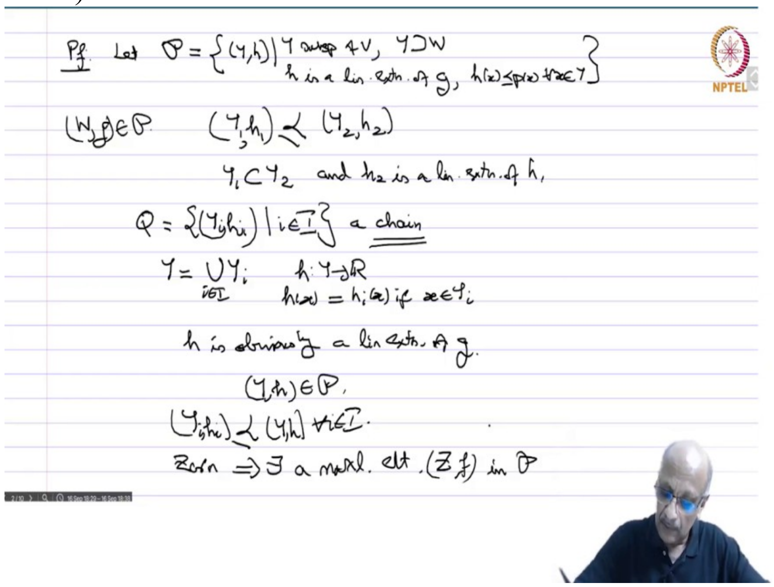

Proof. Let $P=\{(Y,h): Y\, is\, a\, subspace\, of\, V, Y\supset W, h\, is\, a\, linear\, extension\, of\, g\, with\, h(x)\le P(x)\, for\, all\, x\in Y\}$. P is non empty because you already have $(W,g)\in P$ W.

Now we are going to define a partial order on P. Ee say $(Y_1,h_1)\le(Y_2,h_2)$ if Y_1 is contained Y_2 and h_2 is linear extension of h_1. This defines a partial order on the space. We now take $Q=\{(Y_i,h_i): i\in I\}$ to be a chain. What is a chain? A chain means any two elements are comparable i.e., if (Y_1,h_1) and (Y_2,h_2) are in Q then either Y_1 is contained inY_2 or Y_2 is contained in Y_1 and corresponding maps are linear extensions. So, any two maps are comparable. Then you define $Y=\cup_{i\in I}Y_i$. This is again a subspace because $Q$is a chain, hence, given any two elements $x,y\in Y$, they belong in some Y_i and Y_j. But, then one of them is bigger than the other. So, both will belong to say Y_i or both to Y_j. So, $x+y$ will again be there. Same happens with the scalar multiplication. So, this is a subspace. Now you define $h: Y\mapsto R$ such that $h(x)=h_i(x)$ if $x\in Y_i$. Again this is well defined because if $x\in Y_i\cap Y_j$, then one of this will be bigger one, because they are comparable. Let $Y_i\subseteq Y_j$. Therefore, $h_i(x)=h_j(x)$. So, this is well defined. Because of the fact that you are dealing with a chain, clearly h is a linear extension of g. Therefore, $(Y,h)\in P$.

So, every chain has an upper bound; because $(Y_i,h_i)\le(Y,h), \forall i\in I$. Therefore, every chain has an upper bound, and therefore, by Zorn's lemma there exists a maximal element $(Z,f)\in P$. So, what is the maximal element? There is no other element which is bigger than this according to this order relation. Thus, we have (Z, f) is a maximal element in P, Z contains W and f is a linear extension of g So these are all the properties.

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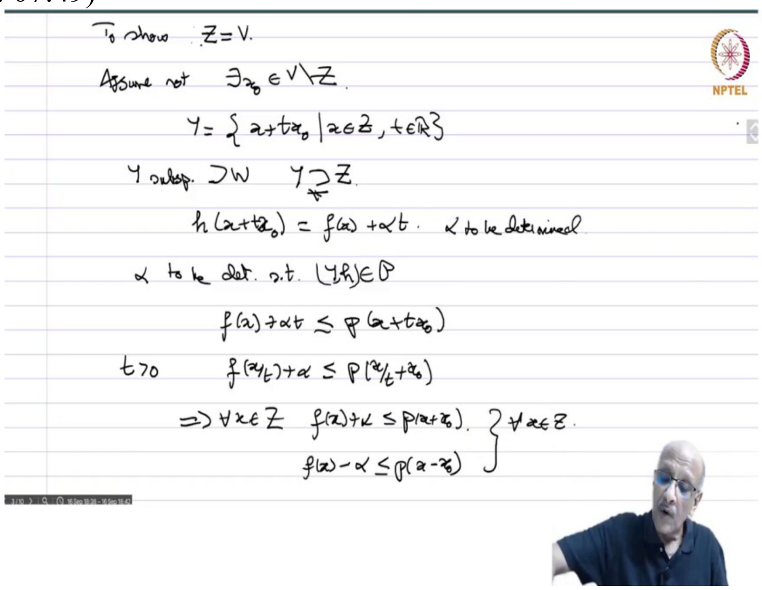

So, to complete the proof we need to show or it is enough for us to show that $Z = V$. Assume the contrary. So, there exists $x_0 \in V \setminus Z$. Now we will define $Y = \{x + t x_0 : x \in Z, t \in R$. So, Y is a subspace which of course contains W ,because if we put $t = 0$, it will contain W. and of course it strictly contains Z.

Now, I am going to define a mapping $h(x + t x_0) = f(x) + \alpha t$, α to be determined. How do I want to determine α ? α to be to be determined such that $(Y, h) \in P$. Then we will have a contradiction because we have contradicted the maximality of (Z, f).

So, we want to see if we can do this. So, Y already contains W and h is also linear. So, we only want to show that $h(x) \leq P(x)$. So, we want to choose α such that $f(x) + \alpha t \leq P(x + t x_0)$.

Now, if $t > 0$, divide through by t, so, you have $f\left(\frac{x}{t}\right) + \alpha \leq P\left(\frac{x}{t} + x_0\right)$. This is true for all x$x$and for all $t > 0$. So, this implies that for all $x \in Z$, you have $f(x) + \alpha \leq P(x + x_0)$. Now, if $t < 0$, you divide by $-t$ and then do the same kind of calculation to get $f(x) - \alpha \leq P(x - x_0)$.

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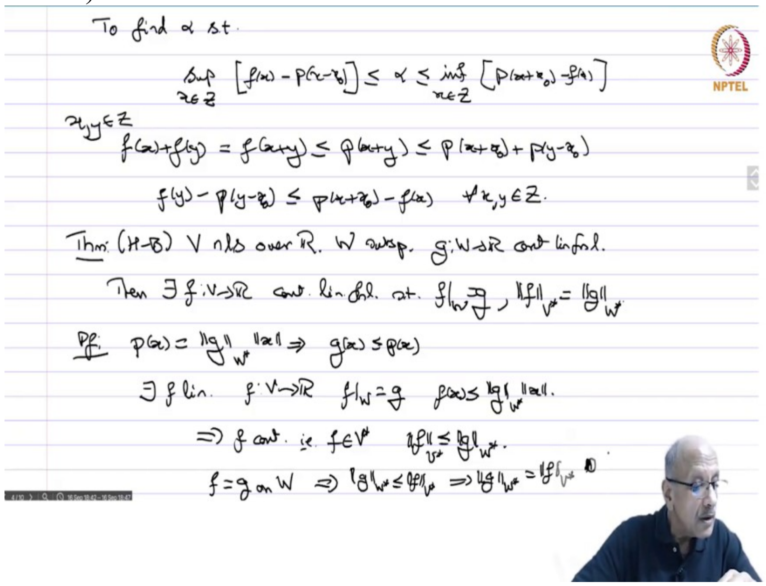

Thus, we have to find α such that

$$\sup_{x\in Z}\left[f(x)-P(x-x_0)\right]\leq\alpha\leq\inf_{x\in Z}\left[f(x)+P(x+x_0)\right].$$

Up to now, we have used only one property of P, namely, positive scalars can go in and out of P Now want to use the other property which is like the triangle inequality kind of thing.

So, we take any $x, y\in Z$, then you have

$$f(x)+f(y)\leq f(x+y)\leq P(x+y)\leq P(x+x_0)+P(y-x_0).$$

This implies $f(y)-P(y-x_0)\leq P(x+x_0)-f(x), \forall x, y\in Z$.

Therefore, the sup of left hand side is always less than the inf of the right hand side. Therefore we can always find an α such that this is true and this completes the proof.

Theorem (Hahn-Banach). Let V be a norm linear space over R and W be a subspace, and $g: W\mapsto R$ be a continuous linear function. Then there exists $f: V\mapsto R$ continuous linear functional such that f restricted to W is g (i.e., it is an extension) and as a bonus we also have that $\|f\|_{V^*}=\|g\|_{W^*}$.

So, we can preserve the norm of the linear functional. So, not only the functional can be extended, we can also do it without the norm going out of control.

Proof. We are going to define $P(x)=\|g\|_{W^*}\|x\|$. Then obviously from the properties of the norm, $P(x+y)\leq P(x)+P(y)$ and $P(\alpha x)=\alpha P(x)$ for α positive. We also have the $g(x)\leq P(x)$ (that is true because, in fact, $|g(x)|\leq\|g\|_{W^*}\|x\|=P(x)$). Therefore, there exists a linear functional $f: V\mapsto R$ (by the previous theorem) such that f restricted to W equal to g, and of course, $f(x) \leq \|g\|_{W^¿} \|x\|$. This implies that f f is continuous and you have that $f \in V^¿$ and $f(x) \leq \|g\|_{W^¿} \|x\|$ for all $x \in V$. So you put $-x$ in place of x. So it is true for minus $-f(x)$also. So it is true for $|f(x)| i.e., \vee f(x) \vee \leq \|g\|_{W^¿} \|x\|$. Therefore, you have that $\|f\|_{V^¿} \leq \|g\|_{W^¿}$. But $f(x) = g(x), \forall x \in W$, therefore, we have that $\|g\|_{W^¿} \leq \|f\|_{V^¿}$ (because you are going to a bigger space, you are going to take the supremum of $¿ f(x) \vee ¿$ over a bigger set and consequently you will have a bigger supremum). So this proves the Hahn-Banach theorem completely.

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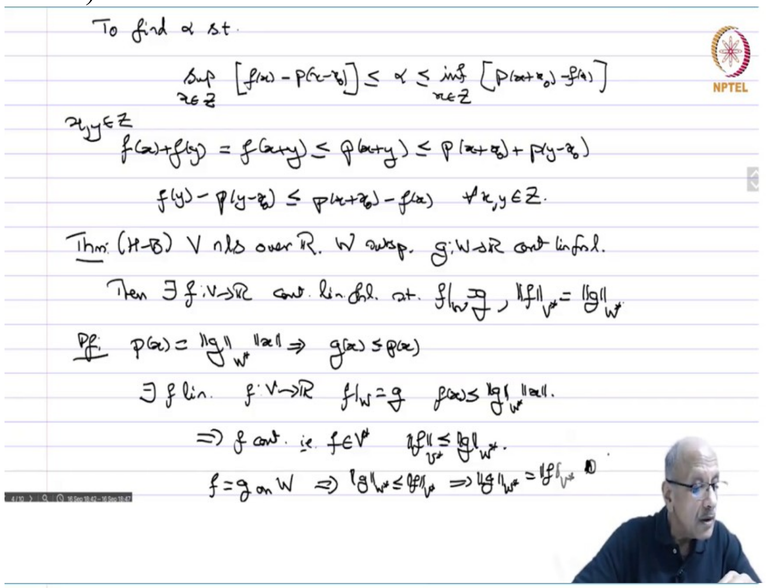

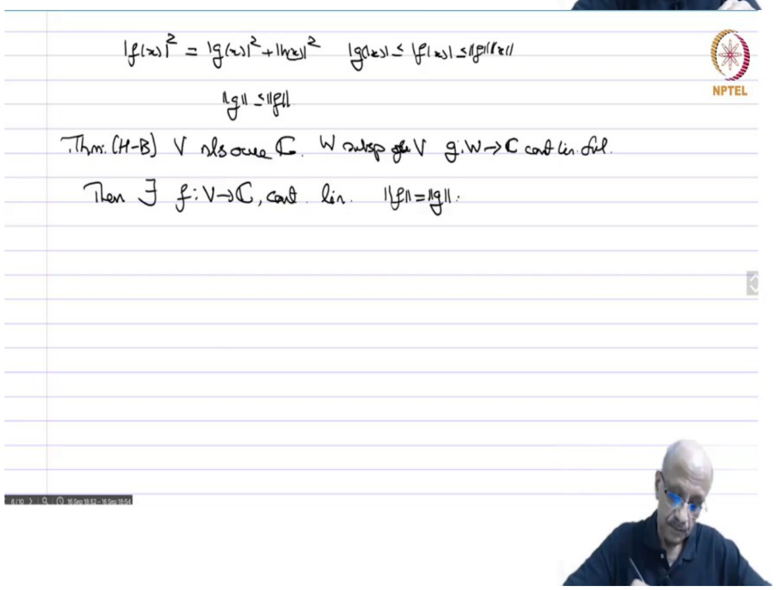

Now, we want to prove the extension theorem also for a complex vector space. Before we do that, we want to look at the anatomy of a continuous linear functional on a complex vector space.

Proposition. Let V be a normed-linear space over C, $f: V \mapsto C$ be a continuous linear functional. Then you write $f = g + ih$, where g, h are the real and imaginary parts of f i.e., $f(x) = g(x) + ih(x)$. Then $f(x) = g(x) - ig(ix), \forall x \in V$ and $\|f\|_{V^*} = \|g\|_{V^*}$.

So, the real part is the major thing and that is why we can extend the previous theorem.

Proof. For $x \in V$, $f(ix) = if(x)$. So, we get $g(ix) + ih(ix) = ig(x) - h(x)$. Therefore, you have $h(x) = -g(ix)$and therefore we are through. So, this proves the first part namely, $f(x)$ can be written entirely in terms of the real part because of the complex linearity.

Now, one can write $|f(x)| = e^{-i\theta} f(x) = f(e^{-i\theta}x) = g(e^{-i\theta}x) + ih(e^{-i\theta}x)$. But, if you look at the left hand side you have $|f(x)|$, which is real. So, automatically h has to be 0 and therefore, you have $|f(x)| \leq \|g\|_{V^{¿}} \vee |x| \vee ¿$ and therefore you have $\|f\| \leq \|g\|_{V^{¿}}$. Now, on the other hand, $|f(x)|^2 = |g(x)|^2 + |h(x)|^2$. Therefore, $|g(x)| \leq |f(x)| \leq \|f\|_{V^{¿}} \vee |x| \vee ¿$ and therefore, $\|g\|_{V^{¿}} \leq \|f\|_{V^{¿}}$. Therefore, you have that $\|f\|_{V^{¿}} = \|g\|_{V^{¿}}$.

So, now, we go on to theorem Hahn-Banach again V norm linear space over C.

Theorem (Hahn-Banach)**.** Let V be a normed-linear space over C and W be a subspace of V. Let $g: W \mapsto C$ g be a continuous linear functional. Then there exists a continuous linear functional $f: V \mapsto C$ with $\|f\| = \|g\|$.

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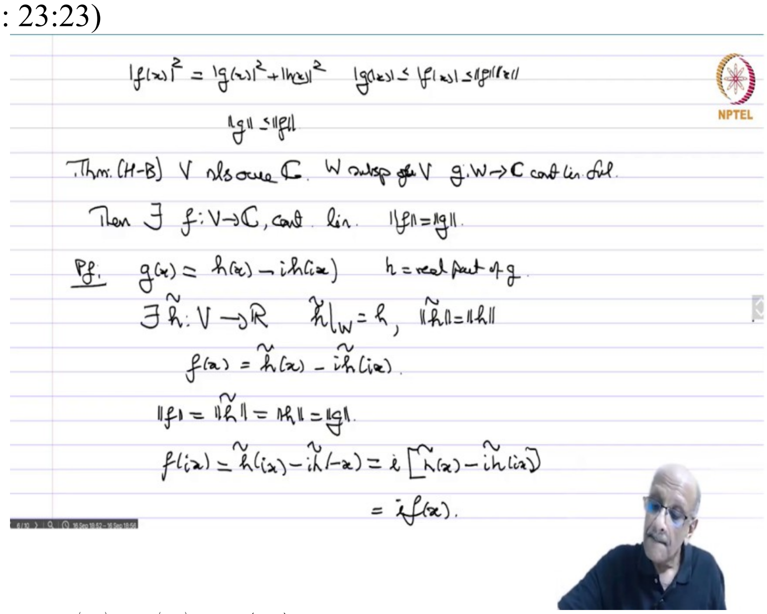

Proof. We will write $g(x) = h(x) - ih(ix)$; $h = ¿$ real part of g, which is a a real linear functional. Then you know that $\|g\| = \|h\|$. So, then by the real Hahn Banach theorem, there exists $\tilde{h}: V \mapsto R$ such that $\tilde{h}$ restricted to W equals h and $\|\tilde{h}\| = ¿ |h| \vee ¿$. These are all considering V as a real vector space. And now I am going to define $f(x) = \tilde{h}(x) - i\tilde{h}(ix)$. Then $\|f\| = \|\tilde{h}\| = \|h\| = \|g\|$. Thus, it is enough to show that f is linear. Vector addition certainly goes through i.e., $f(x+y) = f(x) + f(y)$. Now, for any real scalar α, $f(\alpha x) = \alpha f(x)$ is also clear because $\tilde{h}$ is a real linear functionals. So, we only have to deal with the complex case and by linearity it is enough to check for i. So, $f(ix) = \tilde{h}(ix) - i\tilde{h}(-x) = i ¿$ Therefore, f is complex linear as well and the theorem is proved.

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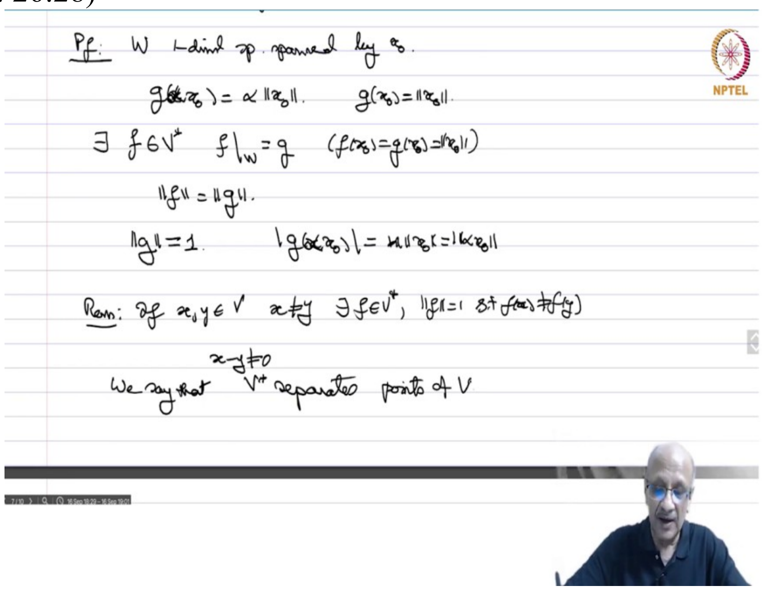

So, in all cases whether it is complex or real vector space, if you have norm linear space and a continuous linear functional on a subspace the Hahn-Banach theorem says we can always extend it continuously to the whole space and you can preserve the norm.

Corollary. Let V be a normed-linear space and $x_0 \in V, x_0 \neq 0$. Then there exists $f \in V^{¿}$ such that $\|f\|=1$ and $f(x_0)=\|x_0\| \neq 0$.

So, this says that the dual space is very rich there are lots of continuous functional. For every $x \in V$ which is non-zero I can produce a special continuous linear functional which has norm 1 and whose value is a certain specific number. So, we can specify whatever we want here and therefore, this really you can construct lots of linear functionals using this particular thing.

Proof. You take W which is a one dimensional space spanned by x_0 and then you define $g(\alpha x_0)=\alpha \vee |x_0| \vee ¿$. In particular, $g(x_0)=\|x_0\|$. Then you can extend g. So, there exists $f \in V^{¿}$ such that f restricted to W is equal to g and therefore, $f(x_0)=g(x_0)=¿|x_0| \vee ¿$ and $\|f\|=¿|g| \vee ¿$. But then, what is $¿|g| \vee ¿$? $\|g\|=1$ because $|g(\alpha x_0)|=|\alpha| \|x_0\|$. So, on the one dimension space W, $g(z)=¿|z| \vee ¿$ for any any $z \in W$ and that means $\|g\|=1$. So, $\|f\|=1$. and that proves this.

Remark. If $x, y \in V$ and $x \neq y$, there exists $f \in V^{¿}$ with $\|f\|=1$ such that $f(x) \neq f(y)$. All you have to do is to look at $x-y (\neq 0)$. And therefore, you can find an f with $\|f\|=1$, $f(x-y)=\|x-y\| \neq 0$.

So, we say that V^* separates points of V. That means, given two distinct points, you can find a continuous linear functional which takes different values at these two points so such that property is called separation of points.

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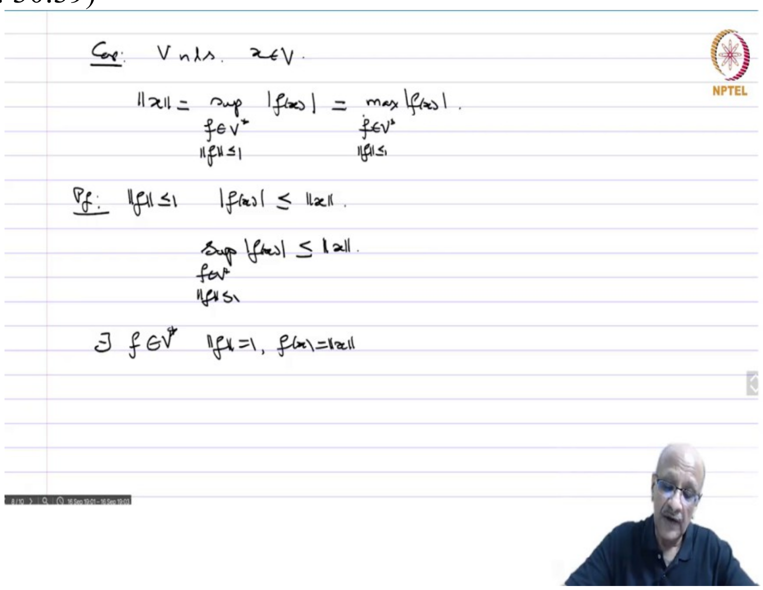

Corollary. Let V be a normed linear space, $x \in V$. Then

$$\|x\| = \sup_{f \in V^*, \|f\| \le 1} |f(x)| = \max_{f \in V^*, \|f\| \le 1} |f(x)|$$

Proof. If $\|f\| \le 1$, then $|f(x)| \le \|x\|$. So $\sup_{f \in V^*, \|f\| \le 1} |f(x)| \le \|x\|$.

On the other hand, we already found there exists $f \in V^*$ and $\|f\| = 1$ and $f(x) = \|x\|$. Therefore, you have that the supremum is in fact equal to the norm. Now, this is the starting point of a very interesting concept which we will see next.