Functional Analysis
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Lecture 2.2
Exercises (Continued)

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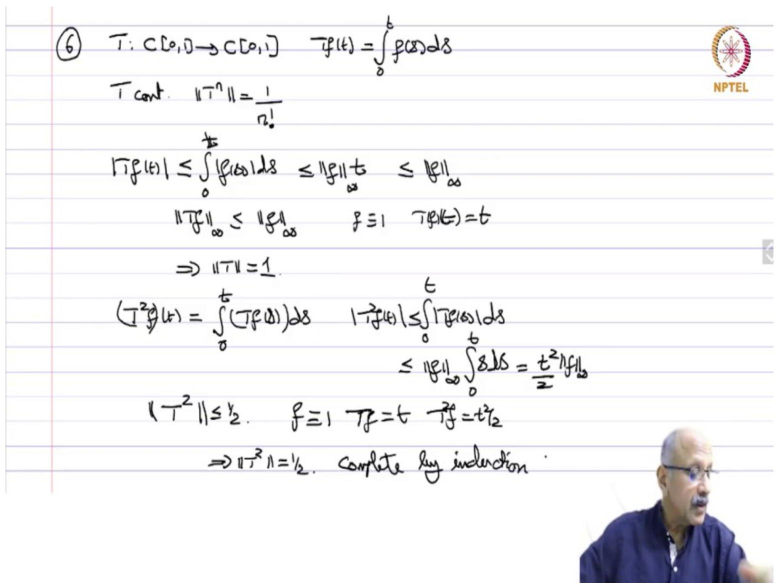

Exercise 6. Now we define $T: C[0,1] \mapsto C[0,1]$ as $T(f)(t) = \int_{[0,t]} f(s)\, ds$ (we are taking the indefinite integral of a continuous function). It is always continuous. So, T definitely maps $C[0,1]$ into $C[0,1]$ and this is a linear map. So, we want to show the$T$ is continuous and also show that $\|T^n\| = \frac{1}{n!}$.

Solution: $|T(f)(t)| \le \int_{[0,t]} |f(s)|\, ds \le \|f\|_\infty t \le \|f\|_\infty$. So, $\|T(f)\|_\infty \le \|f\|_\infty$. Now, if you took $f=1$ then you get $T(f)(t)=t$ and then you see that $\|T(f)\|_\infty = 1$ and therefore you have equality and this implies that $\|T\|=1$. Now $T^2(f)(t) = \int_{[0,t]} T(f)(s)\, ds$. Thus, ¿$T^2(f)(t) \vee \le \int_{[0,t]} |T(f)(s)|\, ds \le \|f\|_\infty \int_{[0,t]} s\, ds = \frac{t^2}{2} \|f\|_\infty$

. Then $\|T^2\| \le \frac{1}{2}$. Again you take $f=1$, then $T(f)$ is nothing but t and $T^2(f)(t) = \frac{t^2}{2}$ and therefore you get in fact that $\|T^2\| = \frac{1}{2}$. Now complete by induction and that will finish.

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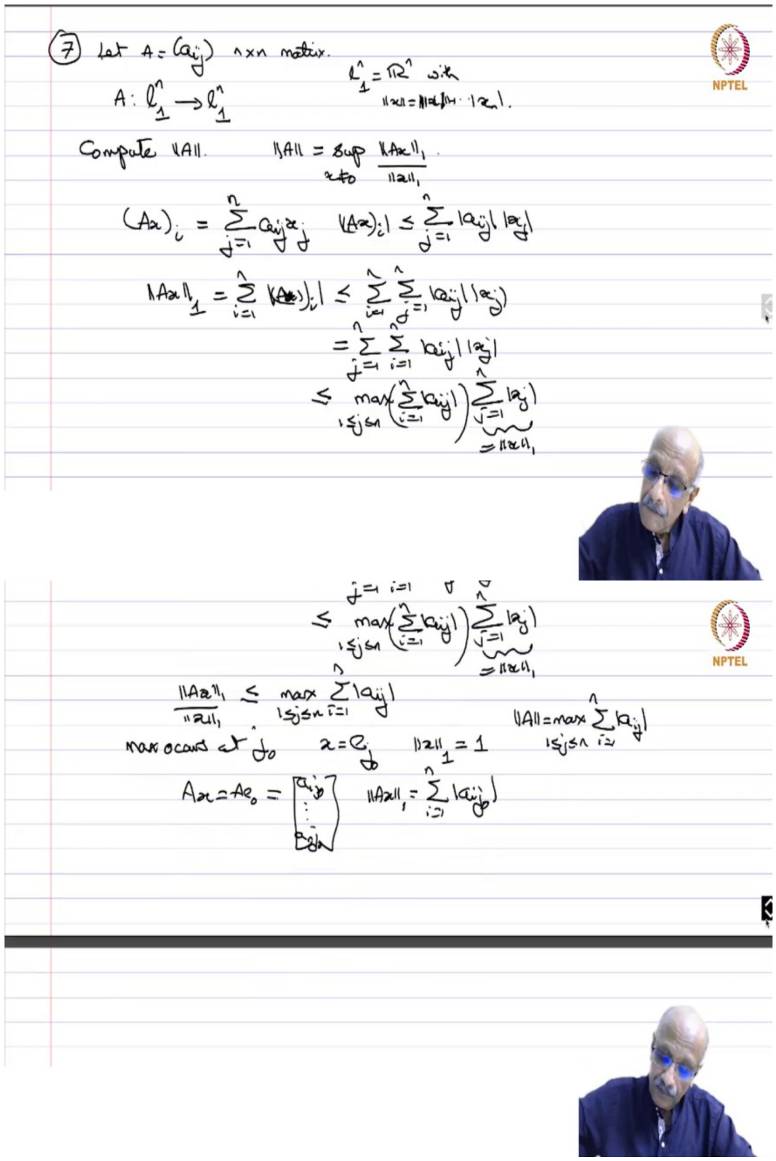

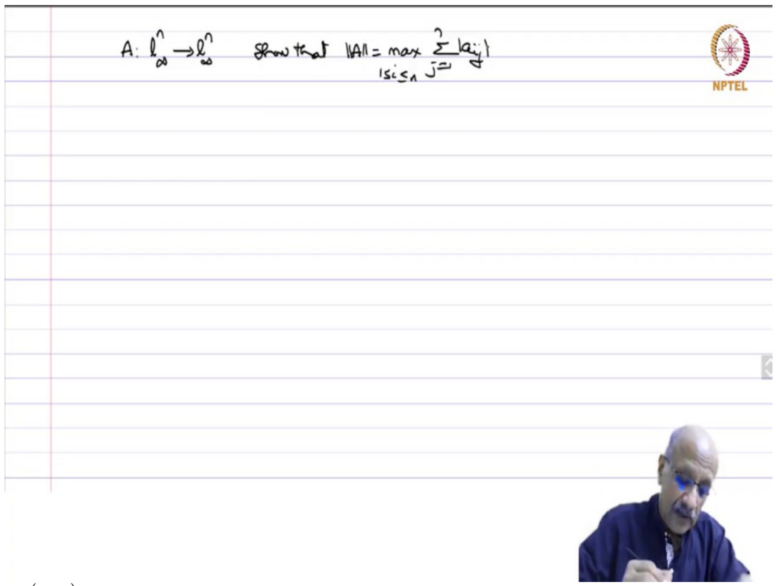

Exercise 7. Let $A=(a_{ij})$ be $N\times N$ matrix. Then A defines a linear transformation on R^N and in finite dimensions all linear transformations are continuous. Let us, take it as $A: l_N^1 \mapsto l_N^1$ (recall l_N^1 is R^N with $¿\vee.\vee|_1$. So, compute $\|A\|$.

<u>*Solution.*</u> Recall that $\|A\| = ¿ x\neq 0 \frac{\|Ax\|_1}{\|x\|_1}$. So, let us compute first

$(Ax)_i = \sum_{j=1,2,\dots N} a_{ij} x_j$which implies $¿(Ax)_i \vee \leq \sum_{j=1,2,\dots N} |a_{ij}||x_j|$.

So, $\|Ax\|_1 = \sum_{i=1,2,\dots N} ¿(Ax)\vee ¿_i \leq \sum_{i=1,2,\dots N} \sum_{j=1,2,\dots N} ¿a_{ij}\vee ¿x_j \vee ¿¿¿$. Let me interchange the order of summation (everything is finite non-negative no problem). So,

$\|Ax\|_1 \leq \sum_{i=1,2,\dots N} \sum_{j=1,2,\dots N} ¿a_{ij}\vee ¿x_j\vee ¿ = \sum_{j=1,2,\dots N} \sum_{i=1,2,\dots N} ¿a_{ij}\vee ¿x_j \vee ¿ \leq \max_{j=1,2\dots,N} \left(\sum_{i=1,2,\dots N} ¿a_{ij}\vee ¿\right) \sum_{j=1,2,\dots N} ¿x_j\vee ¿ = \max_{j=1,2\dots}$

Consequently you get that $\frac{\|Ax\|_1}{\|x\|_1} \leq \max_{j=1,2\dots,N} \left(\sum_{i=1,2,\dots N} ¿a_{ij}\vee ¿\right)¿$ (these are the column sums of the absolute values of the entries). Thus, $\|A\| \leq \max_{j=1,2\dots,N} \left(\sum_{i=1,2,\dots N} ¿a_{ij}\vee ¿\right).¿$

Now, I want to show in fact, this is equal. Assume that maximum occurs at some j_0 then you consider $x=e_{j_0}$,the vector with 1 in j_0 place and 0 elsewhere. Then, $\|x\|_1 = 1$ and what is Ax? $Ax = Ae_{j_0} = [a_{1j_0}, a_{2j_0}, \dots a_{Nj_0}]$ (the j_0 column of A). Therefore, $¿|Ax||_1 = \sum_{i=1,2\dots N} ¿a_{ij_0\vee ¿}¿$ and therefore we have $\|A\| \leq \max_{j=1,2\ldots,N} (\sum_{i=1,2,\ldots N} ¿ a_{ij} \vee ¿) ¿$ and it is actually attained for the vector e_{j_0} and therefore we have $\|A\| = \max_{j=1,2\ldots,N} (\sum_{i=1,2,\ldots N} ¿ a_{ij} \vee ¿) ¿$.

So, in the same way I would like you to try, if A maps from l_∞ to l_∞. Then show then show that $\|A\| = \max_{i=1,2\ldots,N} (\sum_{j=1,2,\ldots N} ¿ a_{ij} \vee ¿) ¿$ (this time you take the row sums of the absolute values of the entries of the matrix and then take the maximum).

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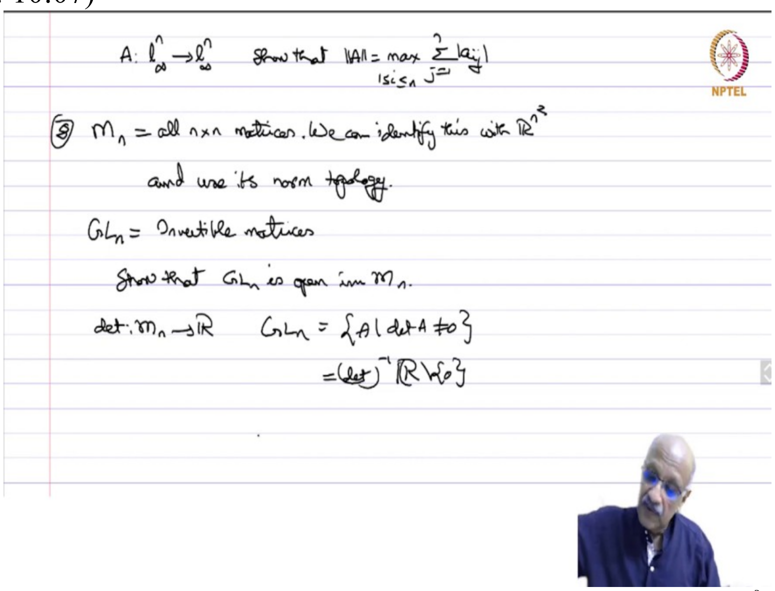

Exercise 8. Let $M_n = ¿$ all $n \times n$ matrices. So, we can identify this with R^{n^2} square and use its topology. Since all norm topologies are equivalent does not matter what you are going to use. So, I am going to identify an element in M_n as a big vector which I string out the rows (or all the columns) and I get n^2 dimensional vector. Let $GL_n = ¿$ invertible matrices. Show that GL_n is open in M_n. (This almost immediate. There is nothing to do. You take the function determinant $det : M_n \mapsto R$. Then this is a continuous function because determinant is nothing but a polynomial in all the variables and therefore it is a continuous function and then $GL_n = \{A : det\, A \neq 0\} = (det)^{-1} (R \setminus \{0\}) ¿$this the inverse image of the open set $R \setminus \{0\}$ with respect to the determinant map). So, inverse image of an open set is open under continuous map and therefore GL_n has to be open and we will do a little more complicated version of this. For that I need to do another exercise.

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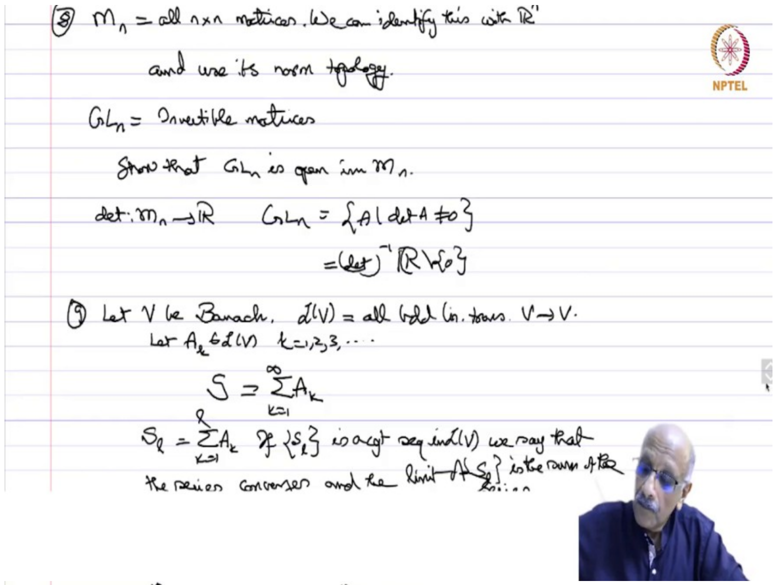

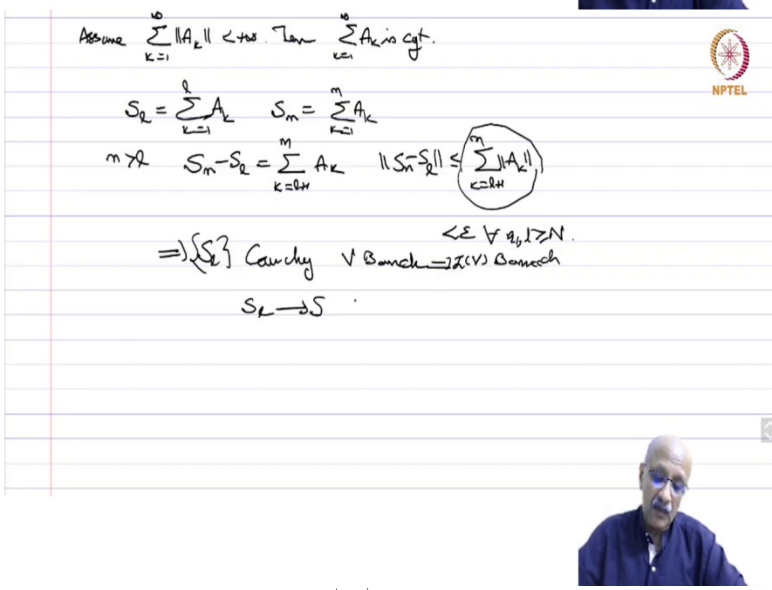

Exercise 9. Let V be Banach and $L(V) =$ ¿ all bounded linear or continuous linear transformations V to V. So $L(V)$ is also a Banach space. Let $A_k \in L(V), k = 1,2,3 \ldots . \infty$. We want to give a meaning to $S = \sum_{k=1,2,\ldots\infty} A_k$. What do you mean by an infinite series? You take $S_l = \sum_{k=1,2,\ldots l} A_k$. So (S_l) is a sequence of partial sums. So, if (S_l) is a convergent sequence in $L(V)$ we say that the series converges and the limit of (S_l) is the sum of the series.

The exercise is, assumed $\sum_{k=1,2,\ldots\infty}$ ¿ $|A_k| \vee$ ¿ $< \infty$ ¿ then $\sum_{k=1,2,\ldots\infty} A_k$ is convergent.

__Solution.__ Let us take $S_l = \sum_{k=1,2,\ldots l} A_k, S_m = \sum_{k=1,2,\ldots m} A_k$. Let us assume $m > l$. So, $S_m - S_l = \sum_{k=l,\ldots,m} A_k$ and therefore, $\|S_m - S_l\| \leq \sum_{k=l,\ldots,m} ¿ \vee A_k \vee ¿$. But $\sum_{k=1,2,\ldots,\infty} ¿ \vee A_k \vee ¿$. is a convergent series. So, by the Cauchy criterion of convergent series, $\sum_{k=l,\ldots,m} ¿ \vee A_k \vee ¿$ can be made less than ϵ for all m, l large enough and consequently (S_l) is Cauchy. Now, V is Banach implies $L(V)$ is also Banach and therefore, (S_l) converges to S and that is called the sum of the convergent series.

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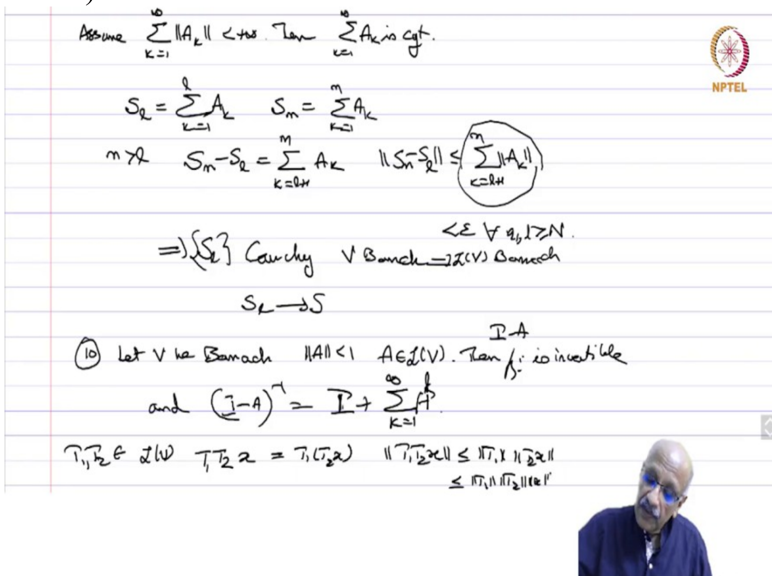

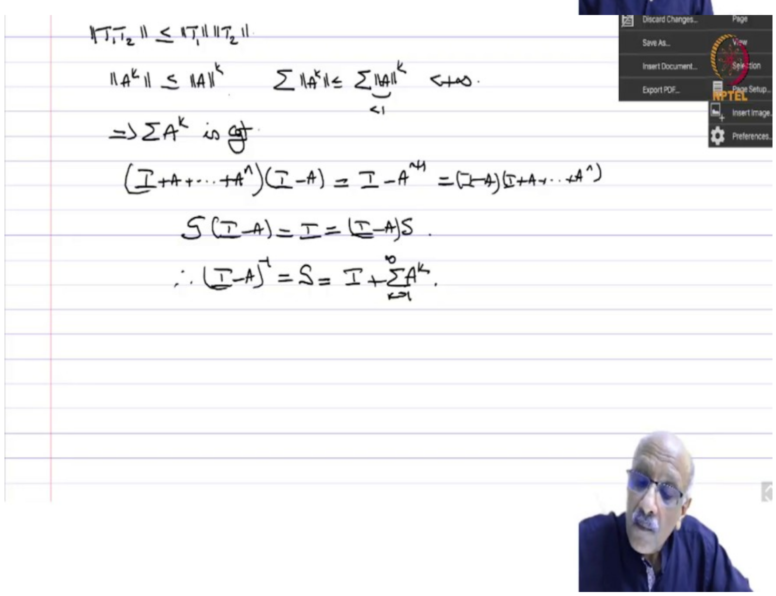

Exercise 10. Let V be Banach and let $\|A\|<1$ for some $A\in L(V)$. Then $I-A$ is invertible and $(I-A)^{-1}=I+\sum_{k=1,2,\ldots,\infty} A^k$. So, this is like if $|x|<1$ then $(1-x)^{-1}=I+\sum_{k=1,2,\ldots,\infty} x^k$. So, this is the infinite dimensional operator version of that.

Solution. $T_1,T_2\in L(V)$ then you can compose them, so, $T_1T_2(x)=T_1(T_2(x))$. Therefore, $\|T_1T_2(x)\|\le\|T_1\|\|T_2(x)\|\le\|T_1\|\|T_2\|\|x\|$. Therefore, $\|T_1T_2\|\le\|T_1\|\|T_2\|$. So, $\|A^k\|\le\|A\|^k$.

Therefore, $\left\|\sum_{k=1,2,\ldots,\infty} A^k\right\|\le\sum_{k=1,2,\ldots,\infty}\|A\|^k<\infty$. $\sum_{k=1,2,\ldots,\infty} A^k$ is convergent by the previous exercise. So, now you look at

$$(I+A+\ldots+A^n)(I-A)=I-A^{n+1}=(I-A)(I+A+\ldots+A^n)$$

So, if you $n\to\infty$, you get $S(I-A)=I=(I-A)S$. Therefore, $(I-A)$is invertible and the inverse is given by S which is a limit of this partial sums, which is $I+\sum_{k=1,2,\ldots,\infty} A^k$.

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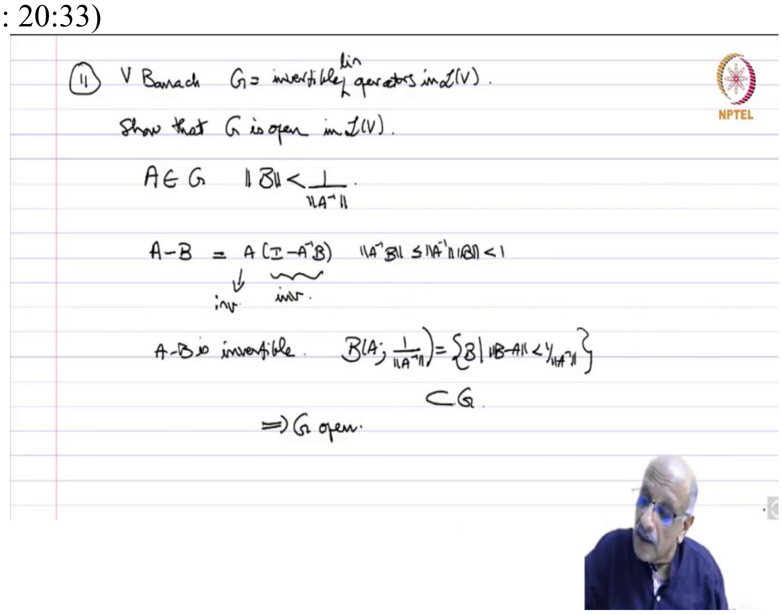

Exercise 11. Let V be Banach and $G=¿$ invertible linear transformations in $L(V)$. Show that G is open.

Solution. This is the infinite dimensional version of the matrix problem which we saw earlier. Now we want to show that G is open and we do not have the notion of a determinant here.

So let us take $A\in G$ and you take B such that $\|B\|<\frac{1}{\|A^{-1}\|}$. Then let us look at

$$A-B=A\left(I-A^{-1}B\right)$$

Now, what about $¿\vee A^{-1}B\vee ¿$? $\left\|A^{-1}B\right\| \leq \vee ¿ A^{-1}\vee ¿ \vee B \vee ¿ < 1$. Thus, $I-A^{-1}B$ is invertible and A is also given to be invertible and therefore $A-B$ is invertible. So this implies that

$$B\left(A;\frac{1}{\left\|A^{-1}\right\|}\right)=\left\{B:\|B-A\|<\frac{1}{\left\|A^{-1}\right\|}\right\}\subseteq G.$$

This implies that G is open.

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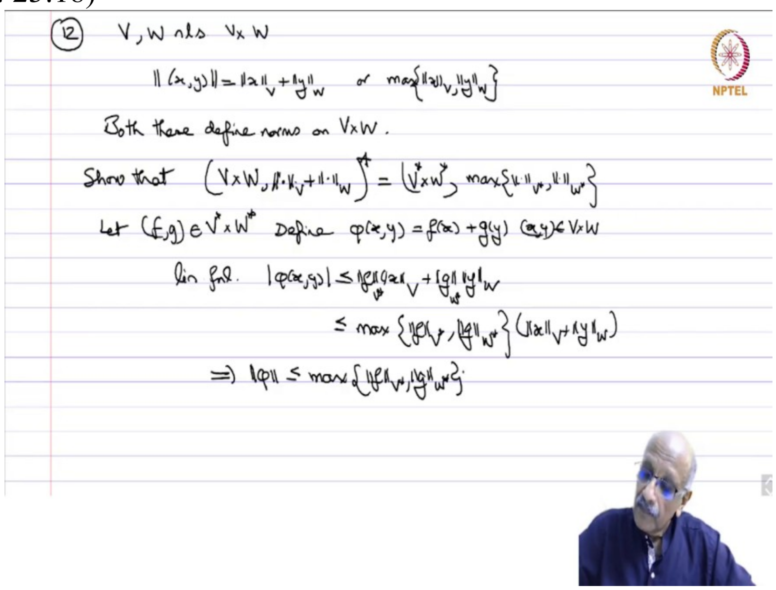

Exaercise 12. The last exercise which I want to discuss with you is the dual of a product space. Let us take V, W, twonorm linear spaces. Then we consider the Cartesian product $V\times W$and I am going to put two norms on this. Let $\|x,y\|_1:=\|x\|_V+\|y\|_W$ or $\|x,y\|_\infty=max\{\|x_V\|,\|y\|_W\}$. Then both these define norms on the $V\times W$.

Show that $\left(V\times W,\|.\|_1\right)^{¿}=\left(V^{¿}\times W^{¿},\|.\|_\infty\right)$.

<u>*Solution.*</u> Let us take $(f,g)\in V^{¿}\times W^{¿}$ and define $\phi(x,y)=f(x)+g(y),(x,y)\in V\times W$.

This is a linear functional and $|\phi(x,y)|=\|f\|_{V^{¿}}\|x\|_V+\|g\|_{W^{¿}}\|y\|_W\leq max\{\|f\|_{V^{¿}},\|g\|_{W^{¿}}\}\left(\|x\|_V+\|y\|_W\right)$.

Therefore, $\|\phi\|\leq max\{\|f\|_{V^{¿}},\|g\|_{W^{¿}}\}$.

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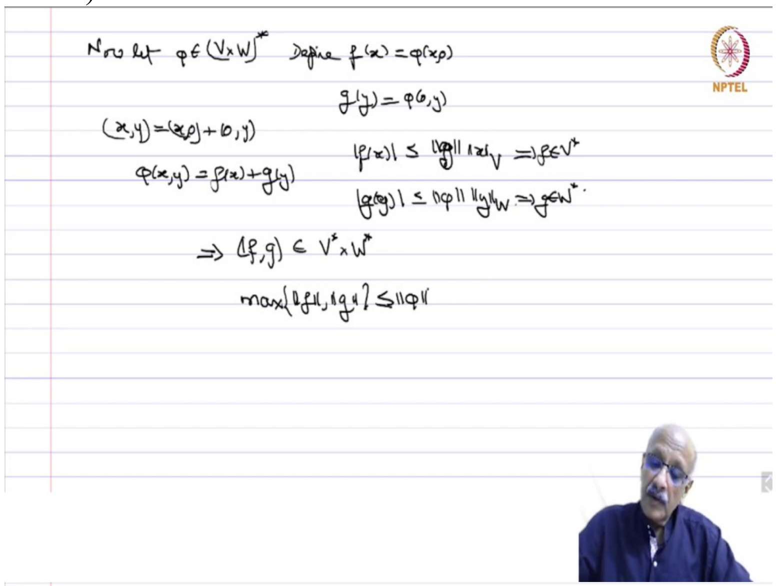

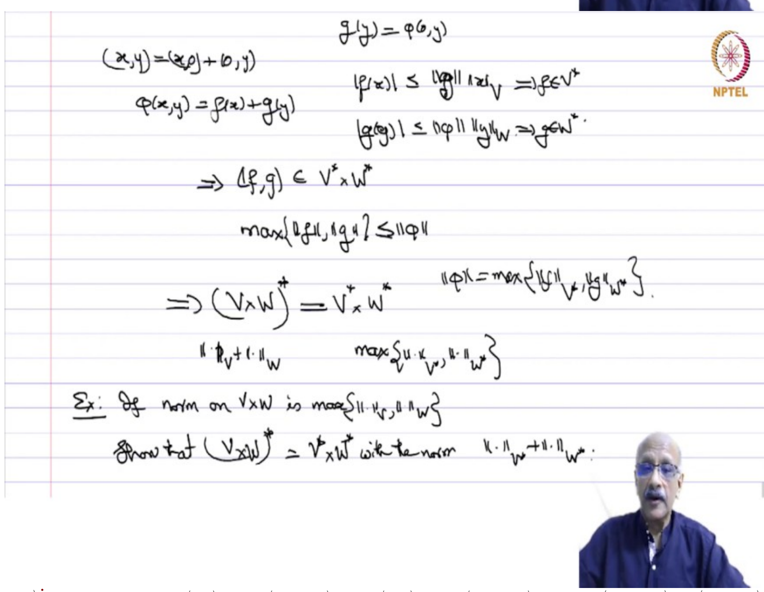

Now, let $\phi \in (V \times W)^{¿}$. Define $f(x) = \phi(x, 0) \wedge g(y) = \phi(0, y)$. So $(x, y) = (x, 0) + (0, y)$. Therefore, $\phi(x, y) = f(x) + g(y)$ (by linearity) and $|f(x)| \leq \|\phi\| \|x\|_V$ and $|g(x)| \leq \|\phi\| \|y\|_W$. This means $f \in V^{¿}$ and $g \in W^{¿}$. Therefore, $(f, g) \in V^{¿} \times W^{¿}$ and $\|(f, g)\|_\infty \leq \vee |\phi| \vee ¿$. Previously we showed $\|\phi\| \leq \|(f, g)\|_\infty$. Thus, $\|\phi\| = \|(f, g)\|_\infty$.

So, $(V \times W, \|.\|_1)^{¿} = (V^{¿} \times W^{¿}, \|.\|_\infty)$.

Exercise. Show that $(V \times W, \|.\|_\infty)^{¿} = (V^{¿} \times W^{¿}, \|.\|_1)$.

one can try to do this yourself. So, I think we will wind up with this.