Functional Analysis
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Lecture No. 74
Exercises - Part 4

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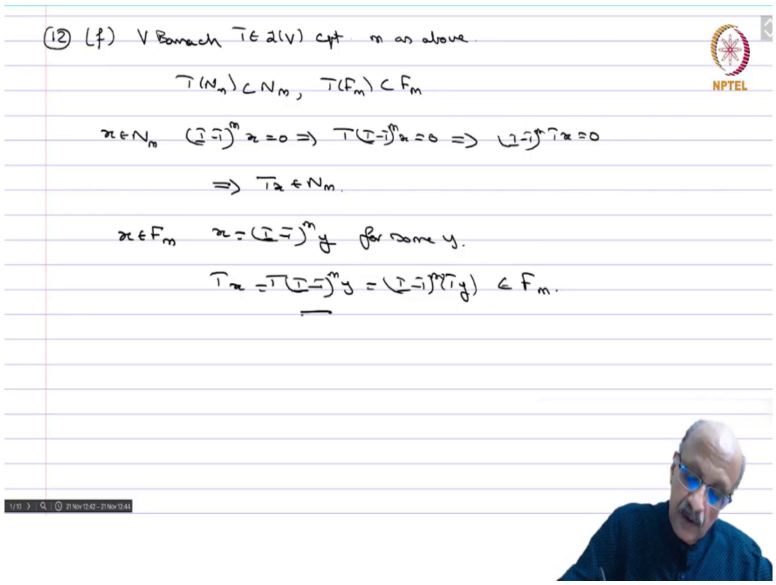

We continue the same exercise. The next exercise I think is 12. So, we did up to **(e)**. So, now **(f)** So, V Banach, $T \in L(V)$ compact, m as above. Then $T\left(N_m\right) \subset N_m$ and $T\left(F_m\right) \subset F_m$.

Solution: So, x is in N_m. So, $(I - T)^m x = 0$, so $T(I - T)^m x = 0$ which implies that $(I - T)^m Tx = 0$. Because T commutes and therefore, this implies the $Tx \in N_m$. So, that proves the first one. Similarly, x is in F_m, $x = (I - T)^m y$ for some y. So, $Tx = T(I - T)^m y$ which is equal to $(I - T)^m Ty$ and therefore, which belongs to F_m again.

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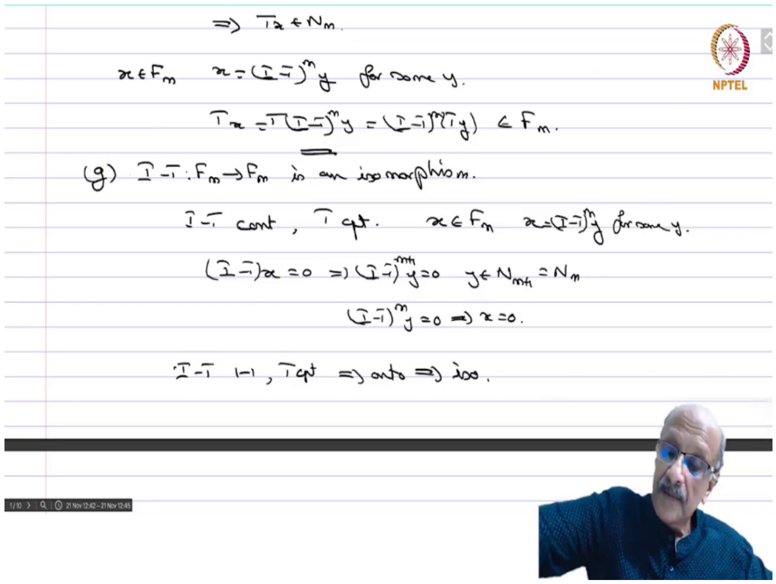

So, that

(g) $I - T: F_m \mapsto F_m$ therefore is an isomorphism.

Solution: So, $I - T$ is continuous T is compact because closed subspace nothing has changed and therefore, let us assume so, $x \in F_m$ means $x = (I - T)^m y$ for some y. So, $(I - T)x = 0$ implies that $(I - T)^{m+1} y = 0$. So, that means $y \in N_{m+1}$ which is the same as N_m. So, $(I - T)^m y = 0$ implies $x = 0$. So, $I - T$ is one-one, T compact. So, this implies onto and by the open mapping theorem this implies $I - T$ is an isomorphism.

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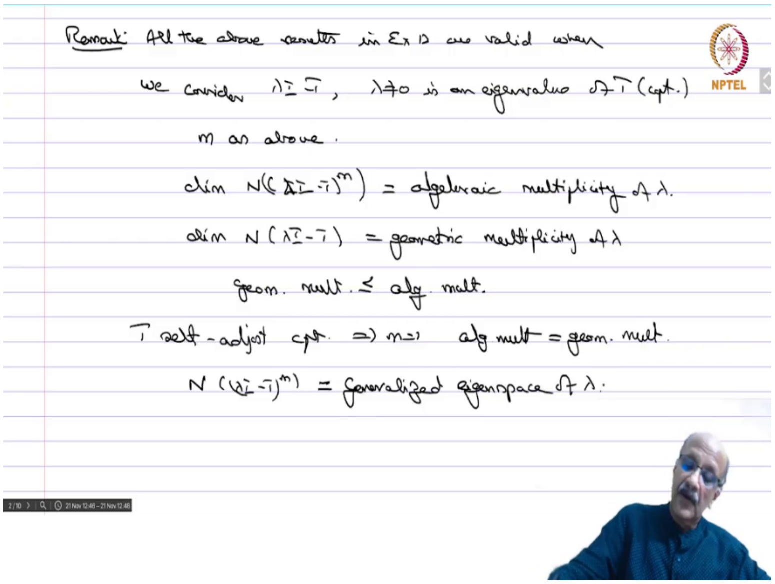

Remark: All the above results in exercise 12 are valid when we consider $\lambda I - T$ where $\lambda \neq 0$ is an eigenvalue of compact of course. Because you can write this as $\lambda\left(I - \frac{T}{\lambda}\right)$ and therefore, all the results which we wrote before they are all true. So, then we have that dimension. So, m as above then dim $N\left((\lambda I - T)^m\right)$ is called the algebraic multiplicity of λ and dim $N(\lambda I - T)$ we know is called the geometric multiplicity. So, number of independent Eigenvectors. So, you have that geometric multiplicity is always less than equal to the algebraic multiplicity. So, the number of eigenvalues because it is an increasing space and therefore, the dimension is always larger, so, you have this. If T is self-adjoint compact, this implies that $m = 1$ therefore, algebraic multiplicity is the same as the geometric multiplicity and $N\left((\lambda I - T)^m\right)$ is called the generalised Eigenspace of λ.

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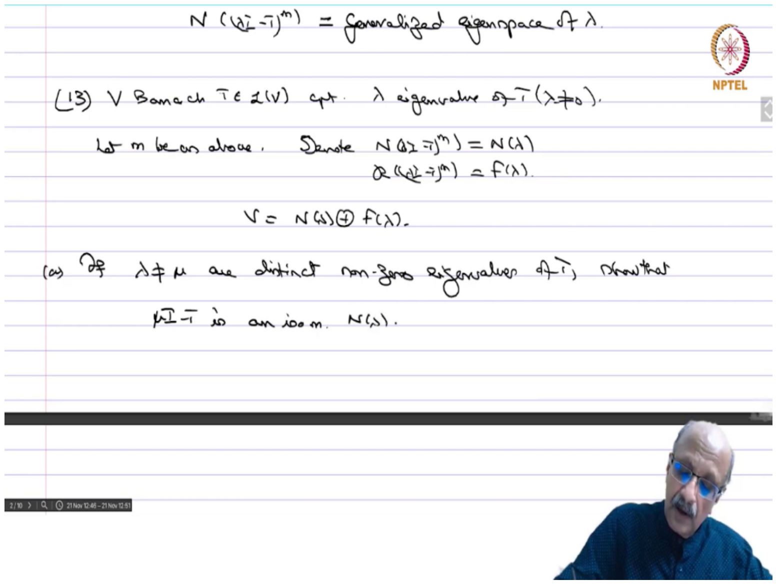

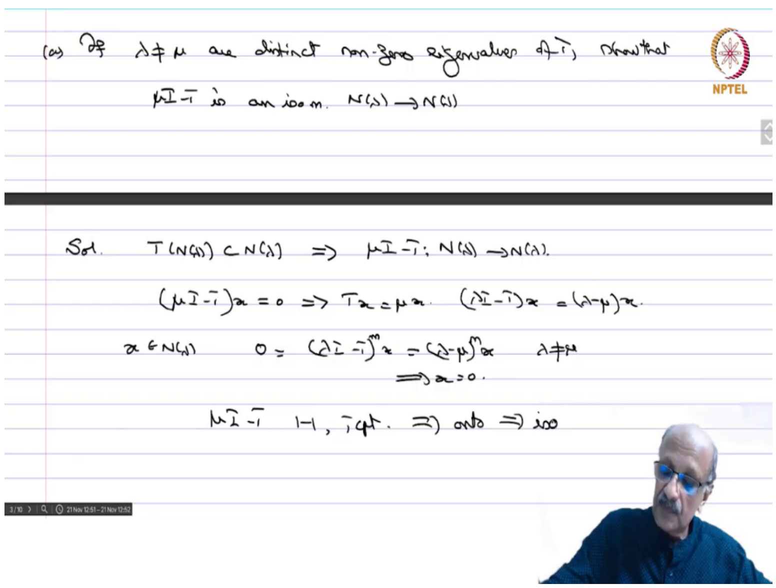

So, now, let us continue so 12 no we are in 13.

Problem 13: So, V Banach, $T \in L(V)$ compact, λ eigenvalue of T, $\lambda \neq 0$. So, then we put an N_m.

Let m be as above and we define or denote $N\left((\lambda I - T)^m\right)$ as $N(\lambda)$ and $R\left((\lambda I - T)^m\right)$ as $F(\lambda)$. So, then we know that $V = N(\lambda) \oplus F(\lambda)$.

(a) If $\lambda \neq \mu$ are distinct nonzero eigenvalues of T, show that $\mu I - T : N(\lambda) \mapsto N(\lambda)$ is an isomorphism.

Solution: $T(N(\lambda))$ we saw is in $N(\lambda)$, we saw that already and therefore, this implies $\mu I - T$ also maps $N(\lambda)$ into $N(\lambda)$. And if $(\mu I - T)x = 0$ this implies that $Tx = \mu x$. So, $(\lambda I - T)x = (\lambda - \mu)x$ and we know that $x \in N(\lambda)$, and therefore $0 = (\lambda I - T)^m x$ which is equal to $(\lambda - \mu)^m x$ and $\lambda \neq \mu$ and therefore this implies that $x = 0$. So, $\mu I - T$ is one-one, T compact and this implies onto and implies it is an isomorphism.

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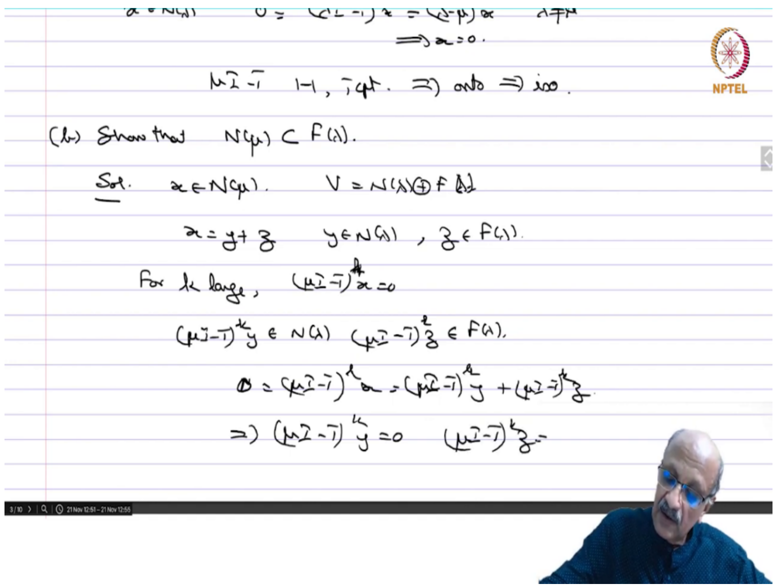

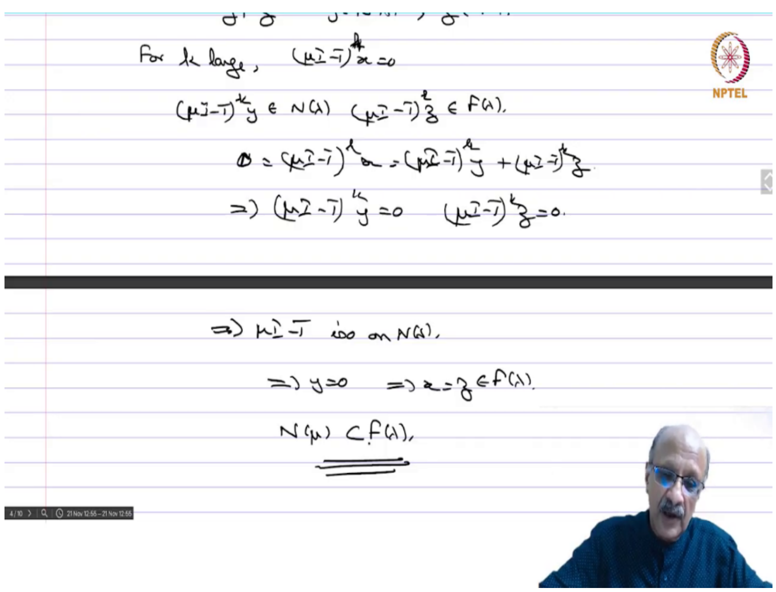

(b) Show that $N(\mu)$ is contained in $F(\lambda)$.

Solution: Let $x \in N(\mu)$ then we know that $V = F(\lambda) \oplus N(\lambda)$ and therefore, x can be written as $y + z, y \in N(\lambda), z \in F(\lambda)$. Then for k large you have that $(\mu I - T)^k x = 0$ that is some k we know because that is the number corresponding to the algebraic multiplicity of μ. But $(\mu I - T)^k y$ continues to live in $N(\lambda)$ and $(\mu I - T)^k z$ continues to live in $F(\lambda)$ and $0 = (\mu I - T)^k x$, which is equal to $(\mu I - T)^k y + (\mu I - T)^k z$. But then this decomposition has to be unique because it is a direct sum. So, this implies that $(\mu I - T)^k y = 0$ and $(\mu I - T)^k z$ is also 0 and since $\mu I - T$ is an isomorphism on $N(\lambda)$ and therefore, you have that $y = 0$. So, this implies $x = z$ which belongs to $F(\lambda)$. So, we have the $N(\mu)$ is contained in $F(\lambda)$.

With this I will wind up the exercises for this chapter. And also, this course has come to an end. I do hope you enjoyed the course and you found it a useful learning experience. All the best.