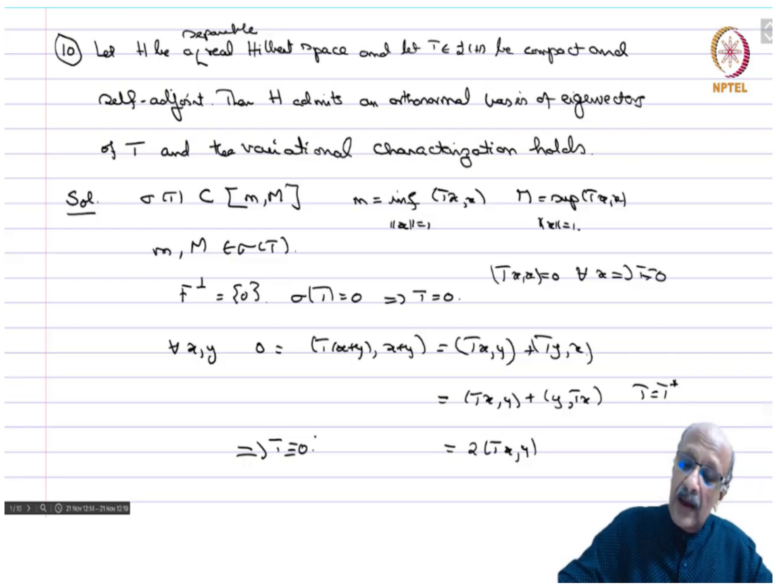
Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematics Sciences
Lecture No. 73
Exercises - Part 3

(Refer Slide Time: 00:18)

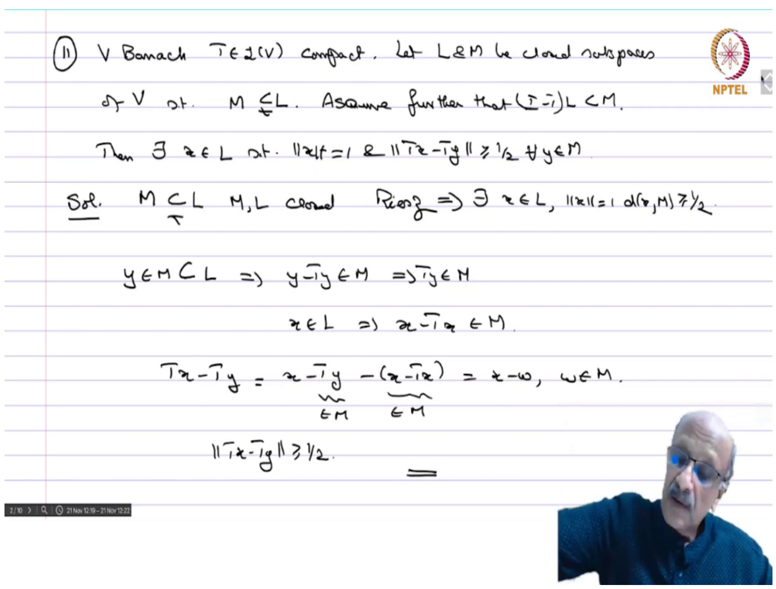


We continue with the exercises.

Problem 10: Let H be a real Hilbert space and let $T \in L(H)$ be compact and self-adjoint and H be a separable real Hilbert space. Then H admits an orthonormal basis of eigenvectors of T and the variational characterization holds. So, if you are dealing with a self-adjoint operator, then of course, it means that everything is true.

Solution: We can show that $\sigma(T)$ in fact is contained in $[m, M]$ where m is the $\langle Tx, x\rangle$ and $M = \langle Tx, x\rangle$. So, this did not require anything about the complex vector spaces and therefore, this you can show, and that m and $M \in \sigma(T)$. So, this part continues to be true even in the real case because we did not use anything. Now, if you go through the proof of the theorem for the structure of the orthonormal basis of eigenvectors, then you remember that there was a space F and we showed that $F^{\perp} = 0$. And this was based on the fact that if you have $\sigma(T) = 0$, then this implies that $T = 0$. And this was because of the fact that if m and M are 0, then $\langle Tx, x\rangle = 0$ for all x and that required the complex structure of the Hilbert space to show that $T = 0$, but if it is self-adjoint, we do not have to do it, because for all x and y, we have $0 = \langle T(x + y), x + y\rangle$. So, that is $\langle Tx, x\rangle$ that is 0, $\langle Ty, y\rangle$ is also 0 and therefore, that is $\langle Tx, y\rangle + \langle Ty, x\rangle$. So, we are showing that $\langle Tx, x\rangle = 0$ for all x implies $T = 0$. So, if we prove this then all the rest of the theory which we have proved so far will work. But $\langle Tx, y\rangle + \langle Ty, x\rangle$ is nothing but $\langle Tx, y\rangle + \langle y, Tx\rangle$ because $T = T^{\star}$ and that is equal to $2\langle Tx, y\rangle$ and this is 0 for all x and y. So, this implies that $T\equiv 0$. So, therefore, the entire theory of which we develop falls compact self-adjoint operators, the existence of orthonormal basis of eigenvectors and the variational characterization, everything goes through and therefore, we have this proof.

(Refer Slide Time: 05:01)

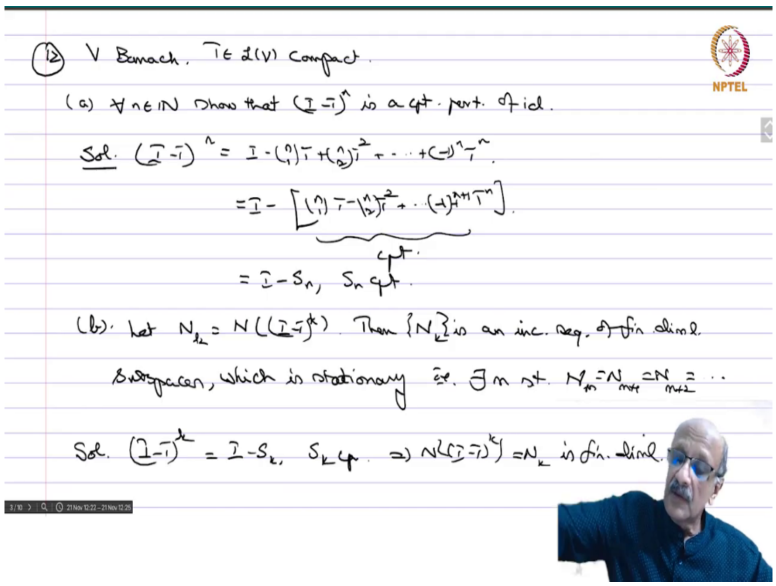


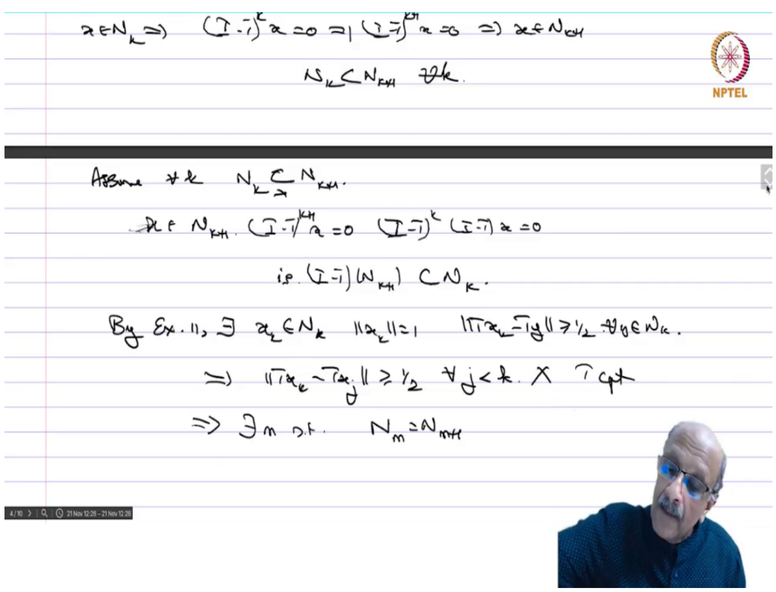
So, now we are going to a series of exercises which are very interesting. And so

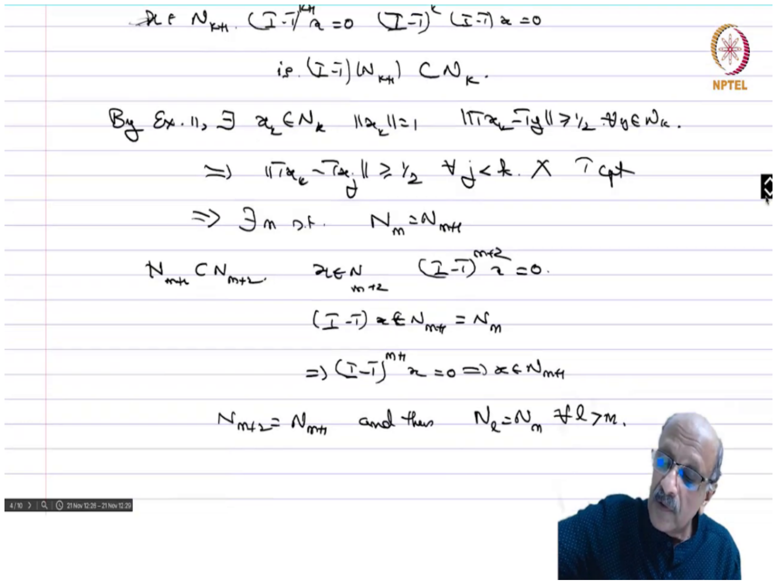
Problem 11: V Banach and $T\in L(V)$. Let, L and M be closed subspaces of V such that you have M is strictly contained in L closed subspace. Also assume further that $(I - T)L\subset M$. So, the bigger space if you apply $(I - T)$ it falls inside this smaller space. Then there exists $x\in L$ such that $||x|| = 1$ and $||Tx - Ty|| \geq \frac{1}{2}$ for all $y\in M$.

Solution: So, this means very much like the Riesz's lemma and in fact we are going to use that. So M is strictly in L, M and L are subspaces so, M, L closed. So, Riesz lemma implies that there exists $x\in L$, $||x|| = 1$ and $d(x, M)\geq 1 - \epsilon$ which we are going to take as equal to $\frac{1}{2}$. So, if you take $y\in M$ that is of course, contained in L and this implies that $y-Ty\in M$ and this implies $Ty\in M$. And if you take an $x\in L$ implies $x-Tx\in M$ as well. So, we have $Tx-Ty=x-Ty-(x-Tx)$, $Ty\in M$, $x-Tx\in M$ and therefore, $x-Ty-(x-Tx)=x-w$, $w\in M$ and therefore, $\|Tx-Ty\|\geq\frac{1}{2}$. So, that proves this problem.

(Refer Slide Time: 07:49)







So, now, we are going to do an important exercise.

Exercise: So, V Banach and $T \in L(V)$ compact.

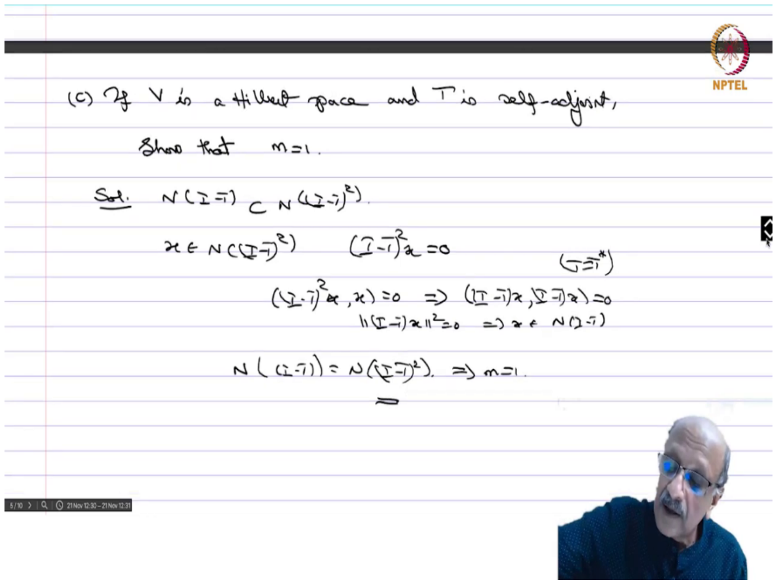
(a) For every $n \in N$ show that $(I - T)^n$ is a compact perturbation of identity.

Solution: So, this a very trivial solution so, $(I - T)^n$ apply the binomial theorem say $I - (n\ 1\ )T + (n\ 2\ )T^2 + \cdots + (-1)^n T^n$ and that is equal to $I - \left[(n\ 1\ )T - (n\ 2\ )T^2 + \cdots + (-1)^{n+1} T^n\right]$. Now, T, T^2 are all compact because T is compact. So, all sum of compact operators is compact. So, this is equal to $I - S_n$, S_n compact. So, that is done.

(b) So, let N_k be the $N\left((I - T)^k\right)$. Then $\{N_k\}$ is an increasing sequence of finite dimensional subspaces which is stationary, that is there exists an m such that $N_m = N_{m+1} = N_{m+2}$ and so on. So, this is what we want to show.

Solution: So, $(I - T)^k = I - S_k$, S_k is compact. So, the $N\left((I - T)^k\right) = N_k$ is finite-dimensional. We have seen this is Riesz-Fredholm theory and also $(I - T)^k x = 0$. So, $x \in N_k$ this implies that $(I - T)^{k+1} x$ is also 0. So, this implies $x \in N_{k+1}$. So, N_k is always contained in N_{k+1} for all k and therefore, it is increasing sequence. So, we want to know that if this is stationary, so, assume for all k, N_k is strictly contained in N_{k+1} that it never is this, now all finite dimensional spaces are closed and you have the strict inequality. So, by Problem 11, you have there exists, we have to check before we can use Problem 11, we have to check one term. So, let $x \in N_{k+1}$. So, $(I - T)^{k+1} x = 0$ and therefore, you have $(I - T)^k (I - T)x = 0$ that is $(I - T)N_{k+1} \subset N_k$. So, the bigger one, so, this is exactly the situation you have for Problem 11. So, by Problem 11, there exists x_k in N_k such that $\|x_k\| = 1$ and $\|Tx - Ty\| \geq \frac{1}{2}$ for all $y \in N_k$. And this means that $\|Tx_j - Tx_k\| \geq \frac{1}{2}$ for all $j \leq k$ and this is a contradiction since T is compact. So, you have a bounded sequence with norm 1 and this implies it cannot have a convergent subsequence and therefore so implies there exists an m such that you have that $N_m = N_{m+1}$. But then we want to show you this constant thereafter. So, let so, N_{m+1} of course is contained in N_{m+2}. So, let $x \in N_{m+2}$. Then $(I - T)^{m+2} x = 0$ that means, $(I - T)(x) \in N_{m+1}$ but N_{m+1} is the same as N_m and therefore, you have $(I - T)^{m+1} x = 0$, because I apply $(I - T)^m$ to $(I - T)(x)$ is 0, that means $x \in N_{m+1}$. Therefore, $N_{m+2} = N_{m+1}$ and thus and $N_l = N_m$ for all $l \geq m$. So, the sequence becomes stationary.

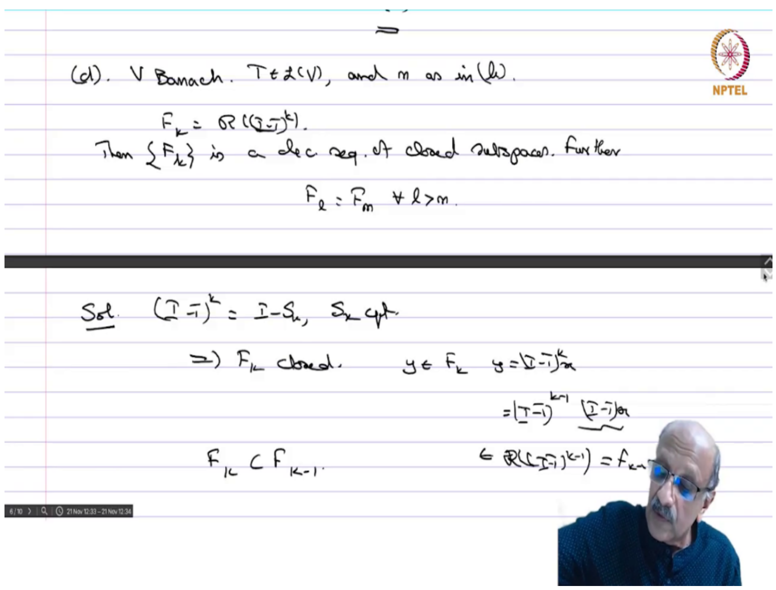
(Refer Slide Time: 14:41)

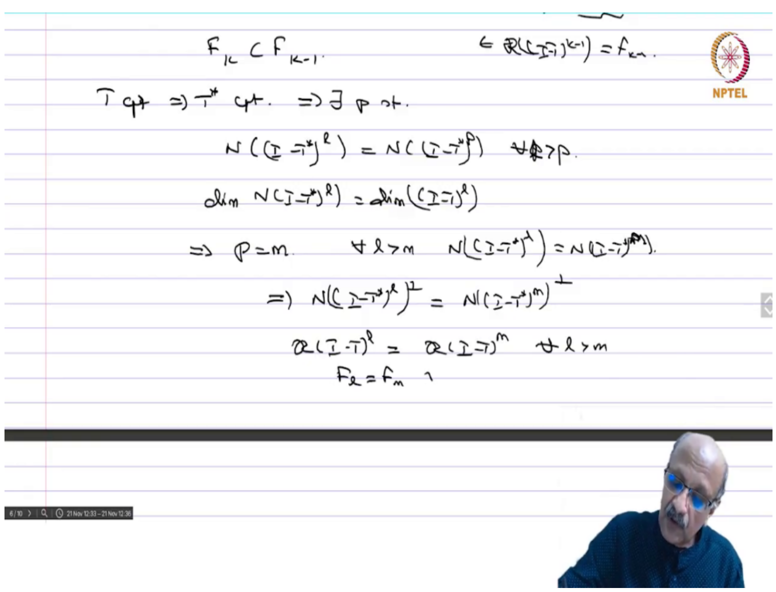


(c) If V is a Hilbert space and T is self adjoint, show that $m = 1$.

Solution: So, the very first step which will happen. So, we have $N((I - T)) \subset N\left((I - T)^2\right)$. So, now let $x \in N\left((I - T)^2\right)$, so, $(I - T)^2 x = 0$, so, $\langle (I - T)^2 x, x\rangle = 0$. Now, you take one $I - T$, $\langle (I - T)x, (I - T)x\rangle = 0$. Since $T = T^*$ when I take the adjoint and that is $\|(I - T)x\|^2 = 0$ implies $x \in N((I - T))$. So, the two are equal. So, $N((I - T)) = N\left((I - T)^2\right)$ and then after all of them have to be equal as we have just seen and therefore, $m = 1$. So, in the self adjoint case it is immediate.

(Refer Slide Time: 16:30)



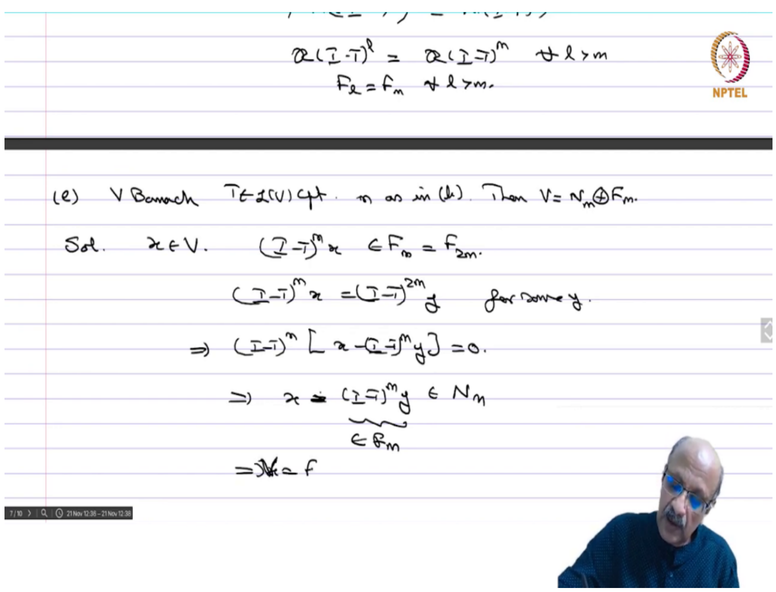


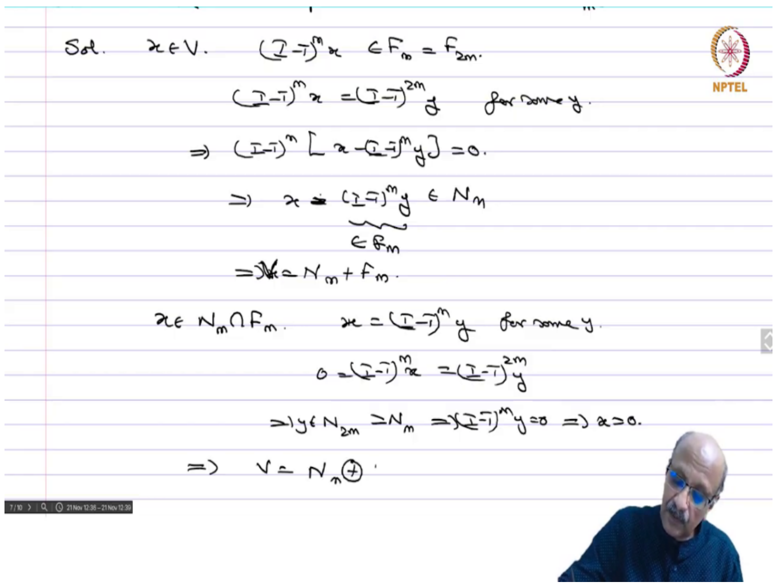
(d) V Banach, $T \in L(V)$ and m as in **(b)**. So, it is the level at which the null space becomes stationary. Define $F_k = (I - T)^k$ then $\{F_k\}$ is the decreasing sequence of closed subspaces. Further $F_l = F_m$ for all $l \geq m$.

Solution: So, $(I - T)^k = I - S_k$, S_k compact. This implies that F_k is closed, the range is always closed. So, if $y \in F_k$, $y = (I - T)^k x$ which is equal to $(I - T)^{k-1}(I - T)x$ and therefore $(I - T)x \in R\left((I - T)^{k-1}\right)$. So, F_k is always contained in F_{k-1}. So, it is a decreasing sequence.

Now, T compact implies $T^{\star}$ is also compact and therefore, there exists a T such that $N\left(\left(I - T^{\star}\right)^{l}\right) = N\left(\left(I - T^{\star}\right)^{p}\right)$ for all $l \geq p$. But dim $N\left(\left(I - T^{\star}\right)^{l}\right) =$ dim $N\left((I - T)^{l}\right)$. So, this is we also know that, because they are adjoined with each other and therefore, the dimensions of the null space are the same as we have already seen and this implies therefore, that $p = m$. So, for all $l \geq m$ you have $N\left(\left(I - T^{\star}\right)^{l}\right) = N\left(\left(I - T^{\star}\right)^{m}\right)$. And now if you take the orthogonal so this implies $\left(N\left(\left(I - T^{\star}\right)^{l}\right)\right)^{\perp} = \left(N\left(\left(I - T^{\star}\right)^{m}\right)\right)^{\perp}$. And that is nothing but $R\left((I - T)^{l}\right) = R\left((I - T)^{m}\right)$ for all $l \geq m$, that is $F_{l} = F_{m}$ for all $l \geq m$.

(Refer Slide Time: 20:26)





(e) V Banach, $T \in L(V)$ compact, m as in **(b)**. Then $V = N_m \oplus F_m$.

Solution: So, let $x \in V$, then $(I - T)^m x \in F_m$ but that is equal to F_{2m} and therefore, this can be written as $(I - T)^m x = (I - T)^{2m} y$ for some y, this implies that $(I - T)^m \left[x - (I - T)^m y\right] = 0$. So, this implies $x = (I - T)^m y \in N_m$ and $(I - T)^m y$ belongs to $R\left((I - T)^m\right)$. So, $(I - T)^m y \in F_m$ and therefore, so, V is therefore equal to $N_m + F_m$. So, let $x \in N_m \cap F_m$ then $x = (I - T)^m y$ for some y and $0 = (I - T)^m x$ which is equal to $(I - T)^{2m} y$. So, implies $y \in N_{2m}$ which is equal to N_m implies $(I - T)^m y = 0$ which implies that $x = 0$. So, this implies that $V = N_m \oplus F_m$.