Functional Analysis
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Lecture No. 72
Exercises - Part 2

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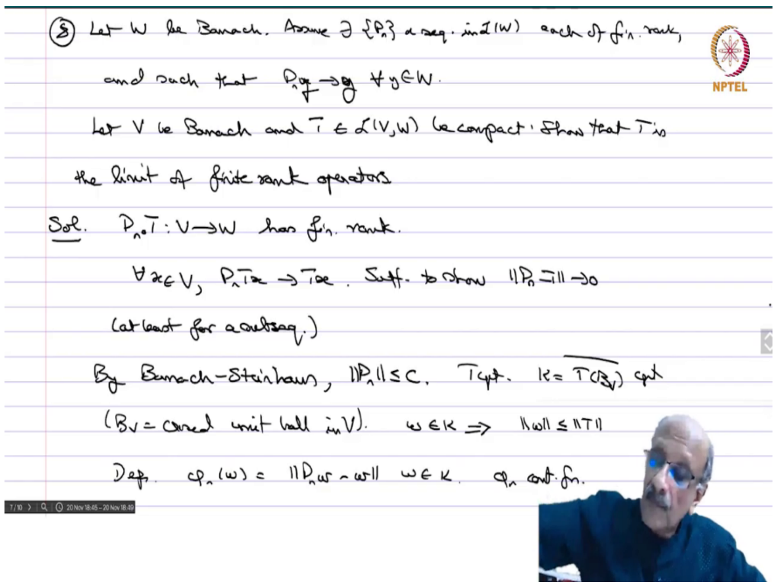

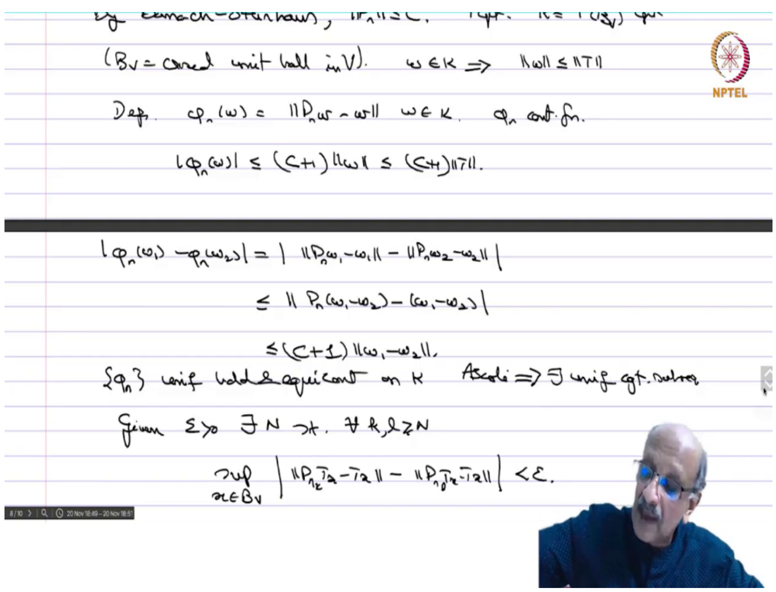

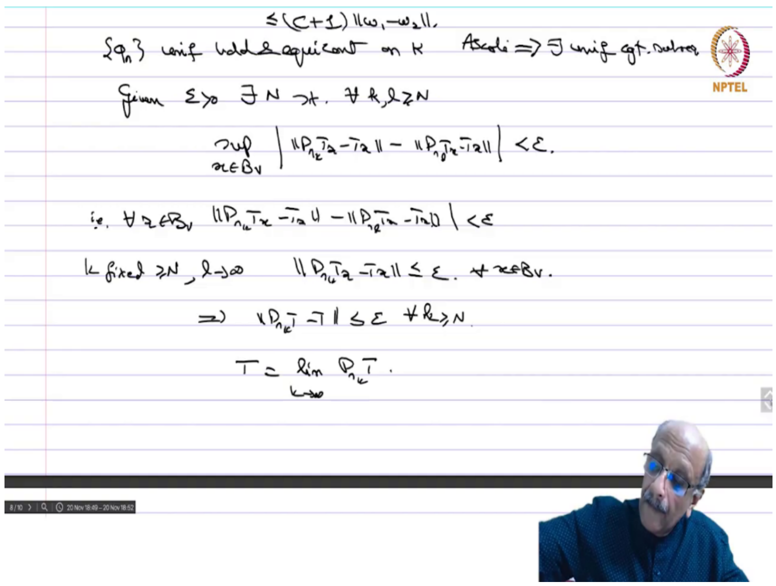

Problem 8: **(a)** So, we studied some conditions under which, when Hilbert spaces were involved, a compact operator can be approximated by a finite rank operator. So, here is another situation where that is possible. Let W be Banach. Assume that exists $\left\{P_n\right\}$ a sequence in $L(W)$, each of finite rank and such that $P_n y \to y$ for all $y \in W$. Let V be Banach, and $T \in L(V, W)$ be compact. Show that T is the limit of finite rank operators.

Solution: So, if you took the obvious one, so $P_n \circ T: V \to W$ has finite rank. And for every $x \in V$, we have $P_n(Tx) \to Tx$ by hypothesis. And therefore, so sufficient to show $\|P_n - T\| \to 0$ at least for a subsequence. We may not be able to show for the whole thing, but it is enough, we all we want is T is a limit to operator of a finite rank. So, if we can show this at least for a finite rank. So, by Banach-Steinhaus, $\|P_n\|$ is uniformly bounded because it converges for each point. So, point wise bounded means uniformly bounded. So, $\|P_n\| \le C$. And then T compact. So, $K = \overline{T(B_V)}$ is compact, where B_V is closed unit ball in V. And if you have $w \in K$, then $\|w\| \le \|T\|$. Because w is the limit of $T(B_V)$, so Tx, x in B_V, so $\|Tx\| \le \|T\|\|x\|$ which is less than $\|T\|$. And in the closure, the same inequality will hold. So, you have that $\|w\| \le \|T\|$. So, define $\phi_n(w) = \|P_n w - w\|$ for w in K. So, ϕ_n is a continuous function and $\left|\phi_n(w)\right| \le (\|P_n\| + \|I\|)\|w\|$ which is $(C + 1)\|w\|$ and this is less than or equal to

$(C + 1)\|T\|$. Therefore, ϕ_n's are uniformly bounded. So, our aim is to use Ascoli's theorem. So, we will now look at $\left|\phi_n\left(w_1\right) - \phi_n(w_2)\right|$ and that is equal to $\left|\|P_n w_1 - w_1\| - \|P_n w_2 - w_2\|\right|$, $|\|x\| - \|y\|| \leq \|x - y\|$. So, $\left|\|P_n w_1 - w_1\| - \|P_n w_2 - w_2\|\right| \leq \|P_n\left(w_1 - w_2\right) - w_1 - w_2\| \leq (C + 1)\|w_1 - w_2\|$. So, $\left\{\phi_n\right\}$ is uniformly bounded and equicontinuous on the compact metric space, so this is a compact metric space, K is a compact metric space. And therefore, Ascoli theorem implies that there exists a uniformly convergent subsequence. So, given ϵ positive, there exists a N such that, for all $k, l \geq N$, we have $\left|\|P_{n_k} Tx - Tx\| - \|P_{n_0} Tx - Tx\|\right| < \epsilon$. So, I am just saying that it is uniformly Cauchy and so in particular, we have this. That is for every $x \in B_V$, you have $\left|\|P_{n_k} Tx - Tx\| - \|P_{n_0} Tx - Tx\|\right| < \epsilon$. Now, you keep k fixed greater than or equal to N and you let l tend to ∞. So, $\|P_{n_k} Tx - Tx\| \to 0$. So, $\|P_{n_k} Tx - Tx\| < \epsilon$ and this is true for all $x \in B_V$. So, this implies that $\|P_{n_k} T - T\| < \epsilon$ for all $k \geq N$. Therefore, we have the $T = P_{n_k} T$. So, we have found a subsequence which converges to this and therefore, this.

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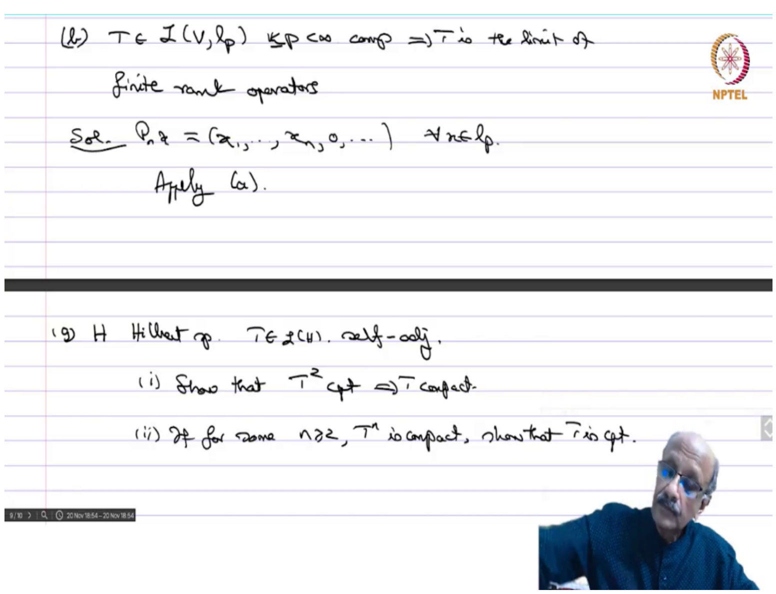

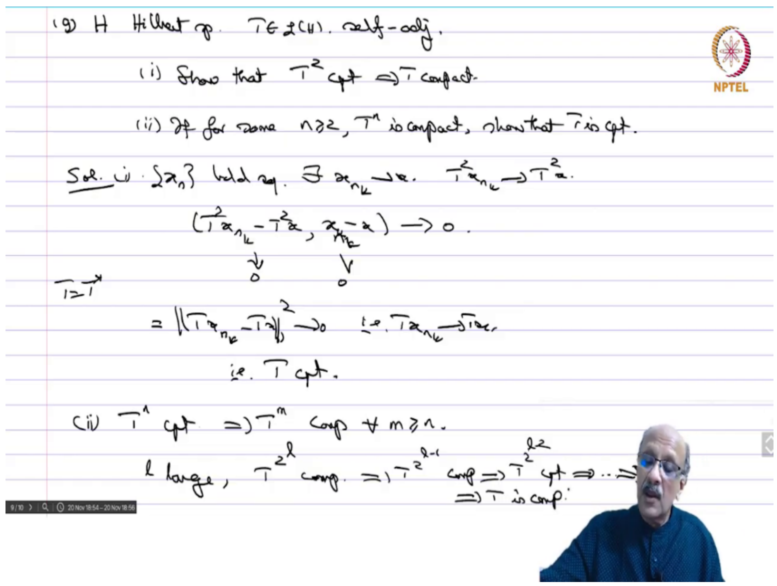

(b) $T \in L(V, l_p)$, $1 \leq p < \infty$ compact implies T is the limit of finite rank operators.

Solution: You put $P_n(x) = (x_1, x_2, \cdot, \cdot, \cdot, x_n, 0, \cdot, \cdot, \cdot)$ for every x in l_p. So, you cut it off. So, then P_n satisfies the conditions of, so apply **(a)**, that is all.

Problem 9: H Hilbert space, $T \in L(H)$ self-adjoint.

(i) Show that T^2 compact implies T compact.

(ii) If for some n, T^n is compact, show that T is compact.

Solution: (i) So, let us take $\{x_n\}$ is a bounded sequence. So, there exists a subsequence $\left\{x_{n_k}\right\}$ which weakly converges to x and you have $T^2 x_{n_k} \to T^2 x$. Therefore, $\langle T^2\left(x_{n_k} - x\right), x_{n_k} - x\rangle$. So, $T^2\left(x_{n_k} - x\right)$ converges in norm, $x_{n_k} - x$ converges weakly and we have seen therefore, $\langle T^2\left(x_{n_k} - x\right), x_{n_k} - x\rangle$ should converge to 0. But what is this? This equal to, I take one T to the other side, so $T = T^{\star}$. So, I get $T\left(x_{n_k}\right) - T(x)$ and if I take one T to that side, I get the same thing and so I get $\|T\left(x_{n_k}\right) - T(x)\|^2 \rightarrow 0$ that is $T\left(x_{n_k}\right) \rightarrow T(x)$, that is T is compact.

So, (ii) is now immediate. Suppose, T^n is compact this implies that T^m is compact for all $m \geq n$. Because composition of compact operators with anything is compact. So, this means for l large, T^{2^l} is compact, this implies $T^{2^{l-1}}$ is compact by what we saw and this implies that $T^{2^{l-2}}$ is compact and so on. This implies that T^2 is compact ultimately and that implies T is compact. So, that completes. So, we will continue the exercises next time.