Functional Analysis
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Lecture No. 71
Exercises - Part 1

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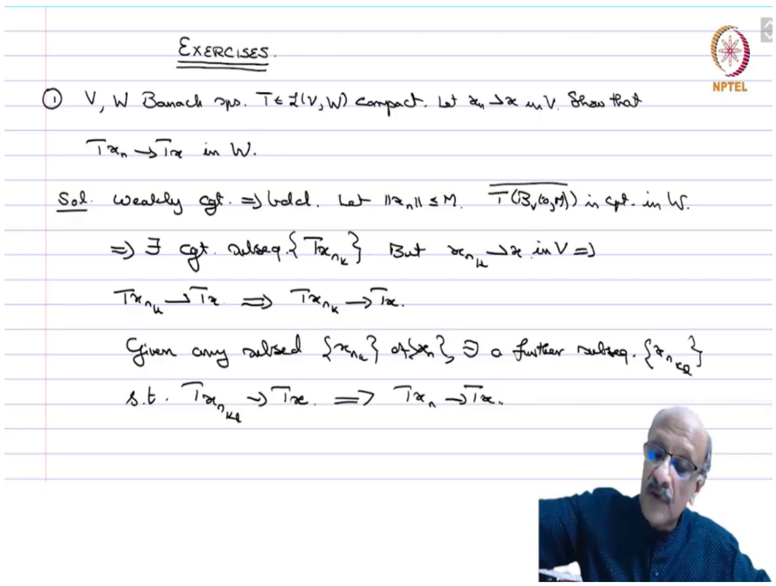

Time for exercises.

Problem 1: V, W Banach spaces and $T \in L(V, W)$ compact. Let $x_n \rightharpoonup x$ in V. Show that $Tx_n \to Tx$ in W. So, a compact operator converts a weakly convergent sequence to a norm convergent sequence in the range. So, that is the powerful property of a compact operator.

Solution: So weakly convergent implies bounded, we know that. So, let $\|x_n\| \leq M$ and then we know the $\overline{T(B_V(0; M))}$ is compact in W. So, implies there exists a convergent subsequence $\left\{Tx_{n_k}\right\}$. But $\{x_n\}$ goes to x weakly, $\{x_{n_k}\}$ also should go to x weakly in V implies T is weakly continuous because of the continuous linear operators. So, $Tx_{n_k} \rightharpoonup Tx$. So, this implies that limit, so $\{Tx_{n_k}\}$ converges in norm and it converges weakly to Tx. Therefore, $Tx_{n_k} \to Tx$. So, this happens for a subsequence. So, given any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ there exists a further subsequence $\{x_{n_{k_l}}\}$ such that $\{Tx_{n_{k_l}}\}$ converges always and the limit will always be Tx. So, the limit is irrespective of the subsequence chosen and I have already mentioned this fact before in a topological space if you have a sequence such that you have any given subsequence has a further subsequence which converges and the limit is independent of the subsequence, the limit is unique then the entire sequence converges. So, this means that $Tx_n \to Tx$, so this proves. So, this you see the topological trick is very very important and very useful and very trivial to prove.

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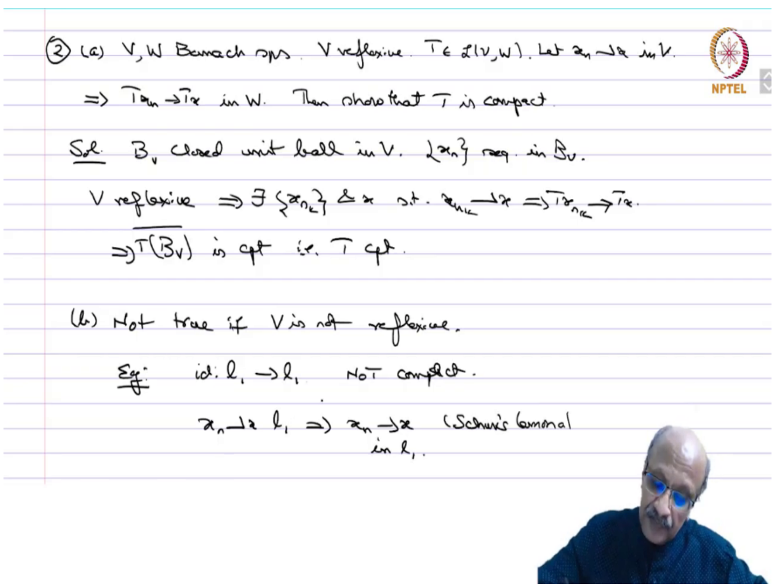

Problem 2: **(a)** So V, W Banach spaces, V reflexive, $T \in L(V, W)$. Let $x_n \rightharpoonup x$ in V, imply $Tx_n \to Tx$ in W. So, we are making this assumption whenever there is a weakly convergent sequence it converts it into a non-convergent sequence. Then show that T is compact. So, this is the converse of the previous problem only we are now assuming that V has to be reflexive.

Solution: So B_V closed unit ball in V. So, $\{x_n\}$ is any sequence in B_V, V reflexive implies there exists $\{x_{n_k}\}$ and x such that $x_{n_k} \rightharpoonup x$ and this implies that $Tx_{n_k} \to Tx$. And therefore, given any sequence $\{x_n\}$, the subsequence converges to Tx. Therefore, $\overline{T(B_V)}$ is compact, that is T is compact.

(b) Not true if V is not reflexive.

Solution: So, let us take an example. So, you take an identity map from l_1 to l_1. So, this is not compact, because l_1 is infinite dimensional. So the identity map is not compact. But if $x_n \rightharpoonup x$ in l_1, this implies $x_n \to x$ in norm, this is Schur's lemma. Therefore, this condition is satisfied, but the mapping is not compact. So, reflexivity is important in this.

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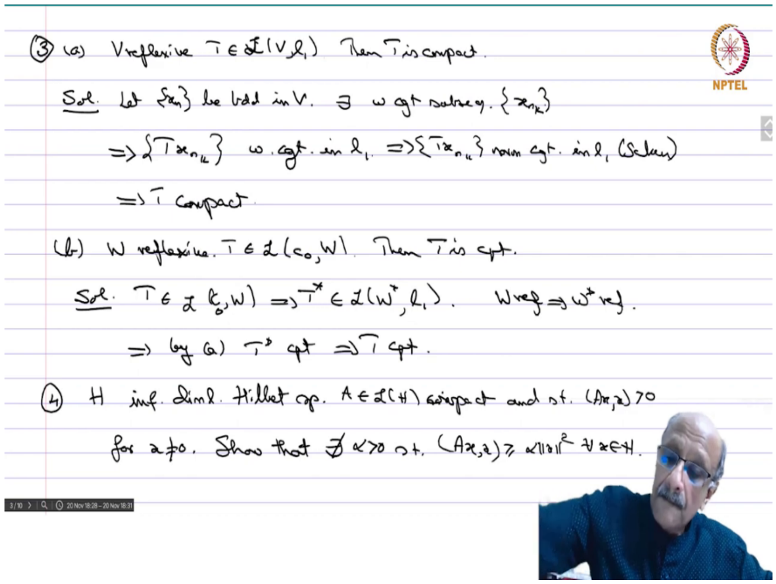

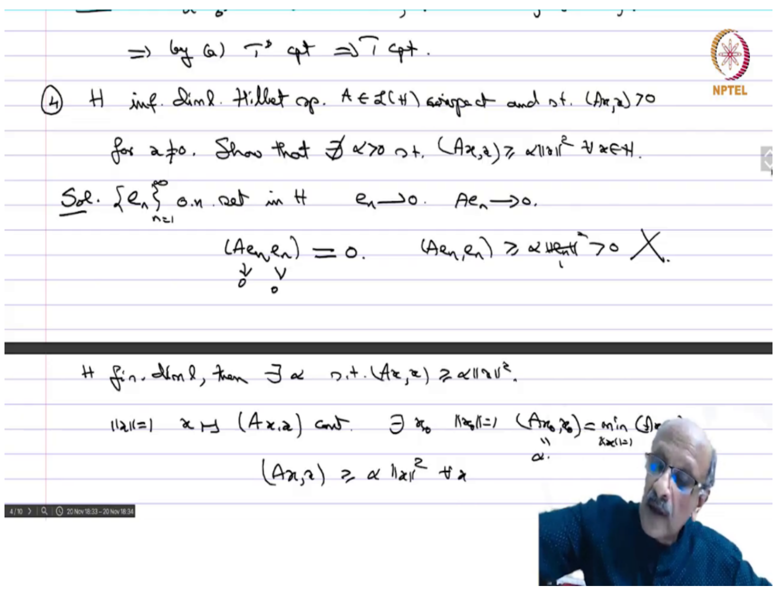

Problem 3: **(a)** V reflexive and $T \in L(V, l_1)$, then T is compact.

Solution: So let $\{x_n\}$ be bounded in V. So, there exists a weakly convergent subsequence. So, $\{x_{n_k}\}$ and this implies that $\{Tx_{n_k}\}$ is weakly convergent in l_1 implies $\{Tx_{n_k}\}$ is norm convergent in l_1 again by Schur's lemma and this implies that T is compact. So, any bounded linear operator from a reflexive space into l_1 is automatically compact.

(b) W reflexive, $T \in L(C_0, W)$. Then T is compact.

Solution: So $T \in L(C_0, W)$, so this implies that $T^\star \in L(W^\star, l_1)$, $C_0^\star$ is l_1 and W reflexive implies $W^\star$ is reflexive. And therefore, by **(a)**, $T^\star$ is compact and then we know that implies that T is compact.

Problem 4: H infinite dimensional Hilbert space. $A \in L(H)$ compact and such that $\langle Ax, x\rangle > 0$ for $x \neq 0$. Show that there does not exist $\alpha > 0$ such that $\langle Ax, x\rangle \geq \alpha \|x\|^2$ for all $x \in H$.

Solution: So we have the $\{e_n\}$ orthonormal set in H, it may or may not be a basis depending whether H is separable or not. But by the Bessel's inequality, we know that $\{e_n\}$ always converges weakly to 0. Therefore, $\{Ae_n\}$ should go strongly or in the norm to 0. So, you have the $Ae_n \to 0$, $e_n \rightharpoonup 0$. And therefore, $\langle Ae_n, e_n\rangle \to 0$ and suppose you have the $\langle Ae_n, e_n\rangle \geq \alpha \|e_n\|^2$, $\|e_n\| = 1$ and $\langle Ae_n, e_n\rangle > 0$ and that is a contradiction. Therefore, you cannot have.

Now, if H were finite dimensional then of course, there exists an α such that $\langle Ax, x\rangle \geq \alpha \|x\|^2$. So, a positive definite operator always has this positive definite matrix. Because if you have $\|x\| = 1$, $x \mapsto \langle Ax, x\rangle$ is continuous and $\|x\| = 1$ unit sphere is compact and therefore, it attains its minimum. So, therefore, there exists x_0, $\|x_0\| = 1$ such that $\langle Ax_0, x_0\rangle = \langle Ax, x\rangle$ and call this as α and then you have $\langle Ax, x\rangle \geq \alpha \|x\|^2$ for all x because you take $\frac{x}{\|x\|}$ and then you get this. So, in finite dimensions this certainly happens and that is why we are trying to see if it happens in infinite dimensions, but it does not happen in infinite dimensions.

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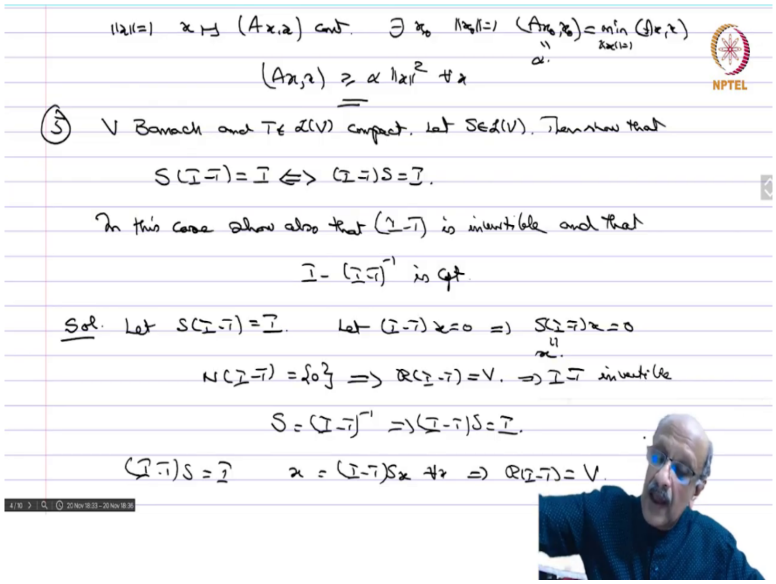

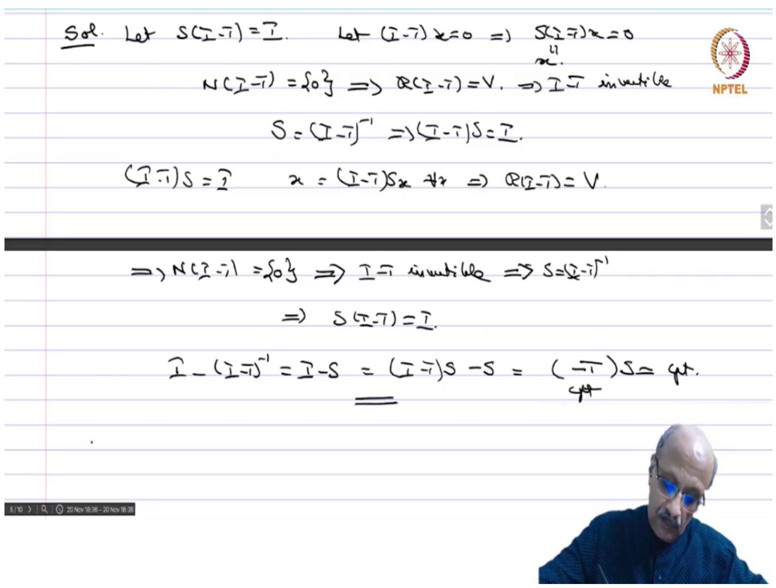

Problem 5: V Banach and $T \in L(V)$ compact. Let $S \in L(V)$. Then show that $S(I-T) = I$ if and only if $(I-T)S = I$. In this case, show also that $(I-T)$ is invertible and that $I-(I-T)^{-1}$ is compact. Because it is almost very, this very trivial.

Solution: So, let $S(I-T) = I$ and then you assume that, let $(I-T)(x) = 0$. So, this implies $S(I-T)(x) = 0$ but $S(I-T)(x) = x$ and therefore, the $N(I-T)$ is $\{0\}$, but then T is compact and therefore, this implies that $R(I-T)$ has to be the whole space V. So, $(I-T)$ is one-one continuous and onto and therefore, $(I-T)$ is invertible and you have $S = (I-T)^{-1}$ and this implies that $(I-T)S = I$. Conversely if $(I-T)S = I$, then $x = (I-T)S(x)$ for all

x, this implies that $R(I - T)$ is the whole space V. Because every x can be written as $(I - T)$ of something. And therefore, $N(I - T)$ has to be 0, one-one if and only if onto because T is compact and therefore this implies that one-one, onto, continuous by open mapping theorem. This means invertible implies $S = (I - T)^{-1}$ and that implies that $S(I - T) = I$. And now, $I - (I - T)^{-1}$ is nothing but $I - S$ and this is equal to $(I - T)S - S$ and that is equal to S into $I - T - I$, which is $(-T)S$ and composition of a compact operator any other, this is compact and therefore, this equal to $(-T)S$ also compact.

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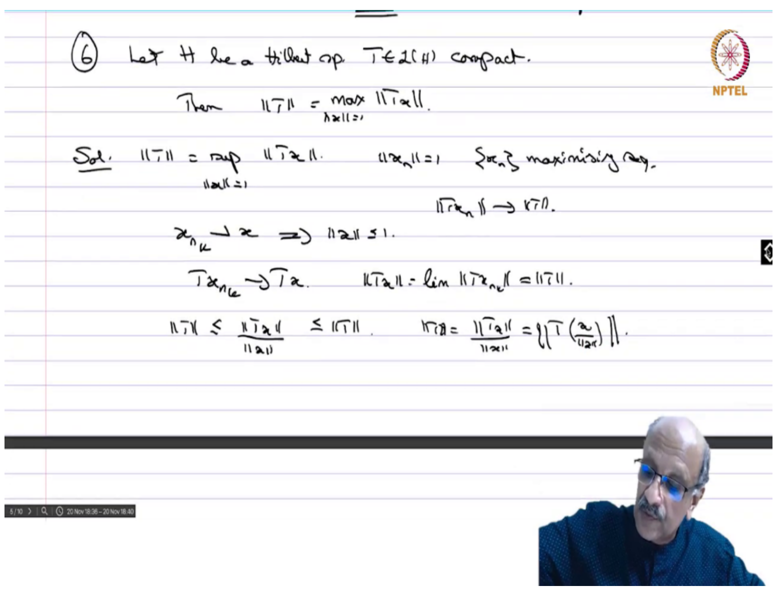

Problem 6: Let H be a Hilbert space and $T \in L(H)$ compact. Then $||T|| = ||Tx||$, we know that it is the supremum and if you have a compact operator, then it is in fact the maximum.

Solution: $||T|| = ||Tx||$. So, let $||x_n|| = 1$, $\{x_n\}$ maximizing sequence. That means $||Tx_n|| \to ||T||$, you can always find it because it is the supremum therefore, you have. Then x_n has norm 1 it is bounded. So, you have a subsequence $\{x_{n_k}\}$ which converges to x and since it is weakly convergent, you have that $||x|| \leq 1$. Then $\{Tx_{n_k}\}$ converges in norm to Tx. Therefore, $||Tx||$ is $||Tx_{n_k}|| = ||T||$. So, and you have that $||x|| \leq 1$. So, $||T||$ is therefore, less than equal $\frac{||Tx||}{||x||}$ because $||x|| \leq 1$. So, I am dividing by something smaller than 1. So, this becomes bigger but then this is always less than or equal to $\|T\|$. Therefore, $\|T\| = \frac{\|Tx\|}{\|x\|}$ which is $\|T\left(\frac{x}{\|x\|}\right)\|$, which has norm 1. And therefore, you have that it is the maximum over the unit sphere, maximum over the unit ball whatever you want.

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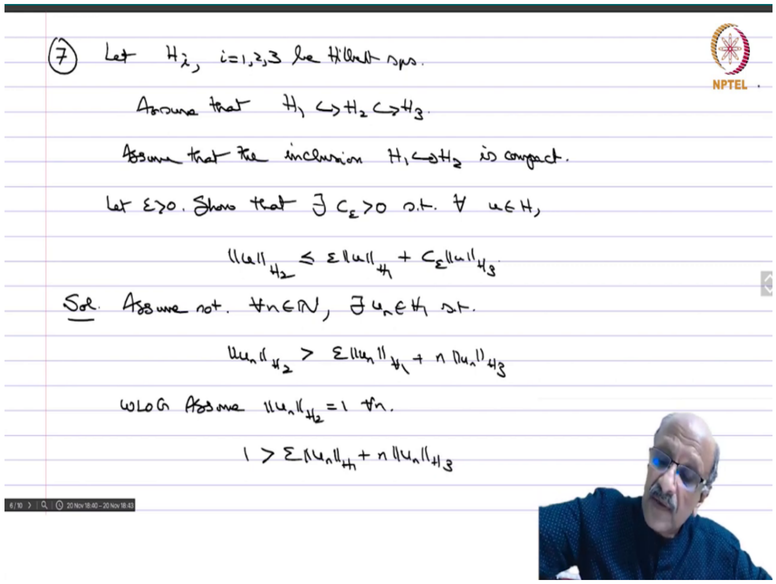

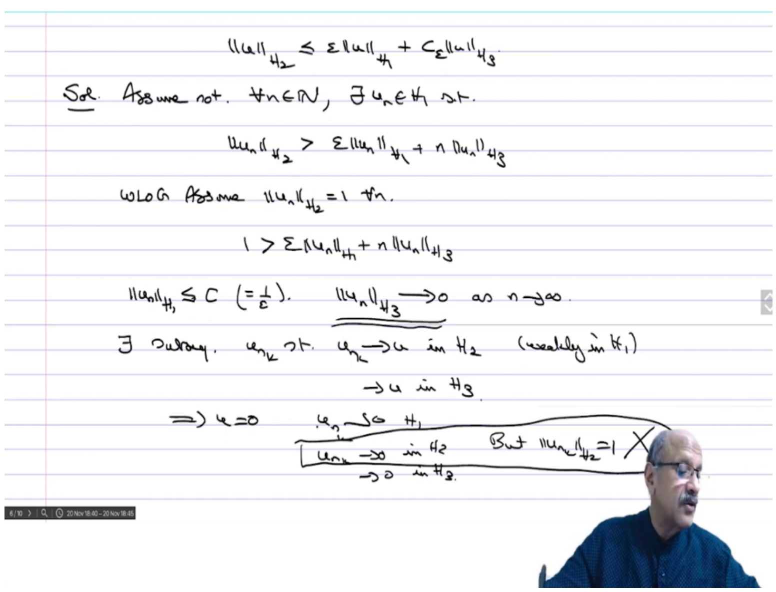

Problem 7: Let H_i, $i = 1, 2, 3$ be Hilbert spaces. Assume that $H_1 \hookrightarrow H_2 \hookrightarrow H_3$. That means, these are subspaces and the inclusion maps are continuous. Assume that the inclusion $H_1 \hookrightarrow H_2$ is

compact. Let ϵ be positive. Show that there exists a C_ϵ which is positive such that for all u in H_1, which is in all the three spaces, we have that

$$\|u\|_{H_1} \leq \epsilon \|u\|_{H_2} + C_\epsilon \|u\|_{H_3}.$$

Such inequalities are very useful in study of partial differential equations and the compactness helps you to get, very small epsilon you can put here and then you compensate it by putting a big constant here for the other function.

Solution: Assume not, then whatever constants you put here you can find vectors which violate this inequality. So, for every $n \in N$, there exists a $u_n \in H_1$ such that $\|u_n\|_{H_2} > \epsilon \|u_n\|_{H_1} + n\|u_n\|_{H_3}$. So, I am taking C_ϵ as n and therefore, for every such n, I will have a violation and therefore, this is true. Now, I can divide throughout by $\|u_n\|_{H_2}$. So, without loss of generality, assume $\|u_n\|_{H_2} = 1$ for all n. So, otherwise you just divide through and that will give you that. So, you have $1 > \epsilon\|u_n\|_{H_1} + n\|u_n\|_{H_3}$. So, this says that $\|u_n\|_{H_1}$ is less than or equal to some constant C which is equal to $\frac{1}{\epsilon}$ if you like, and then $\|u_n\|_{H_3} \leq \frac{1}{n}$, so this goes to 0, as n tends to ∞. So, that exists because the inclusion of $H_1 \hookrightarrow H_2$ is compact. So, we have to use that hypothesis. So, there exists subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ in H_2 and weakly in H_1 and it also therefore, automatically converges to u in H_3 also and but $\|u_{n_k}\|_{H_3} \to 0$, so this implies that $u = 0$ because of this fact. So, you have $u_{n_k} \rightharpoonup 0$ in H_1, $u_{n_k} \to 0$ in H_2, and $u_{n_k} \to 0$ in H_3 as well. But $\|u_{n_k}\|_{H_2} = 1$ and therefore, you have a contradiction to this. So, therefore, there exists such a constant.