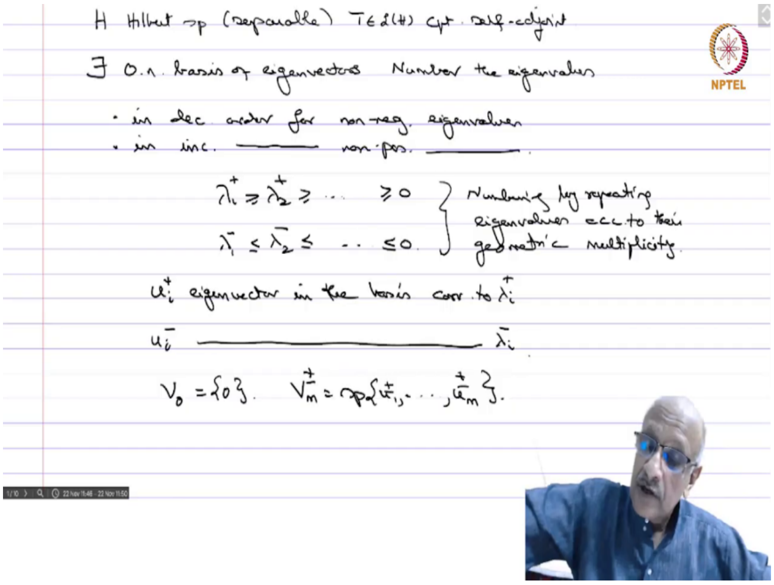
Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences

Lecture No. 70
Eigenvalues of a compact self-adjoint operator

(Refer Slide Time: 00:28)



So, we will now give a variational characterization of the eigenvalues and eigenvectors for a compact self-adjoint operator. So, H Hilbert space which is separable and $T \in L(H)$ compact and self-adjoint. So, you have an orthonormal basis of eigenvectors. So, for each eigenvector choose a corresponding eigenvalue and then number the eigenvalues in decreasing order for non-negative eigenvalues and in increasing order for non-positive eigenvalues.

So, for the result I am going to state below, it does not matter where we put the eigenvalue 0, if at all it is an eigenvalue because it will come last. So, you will have that λ, so we call them

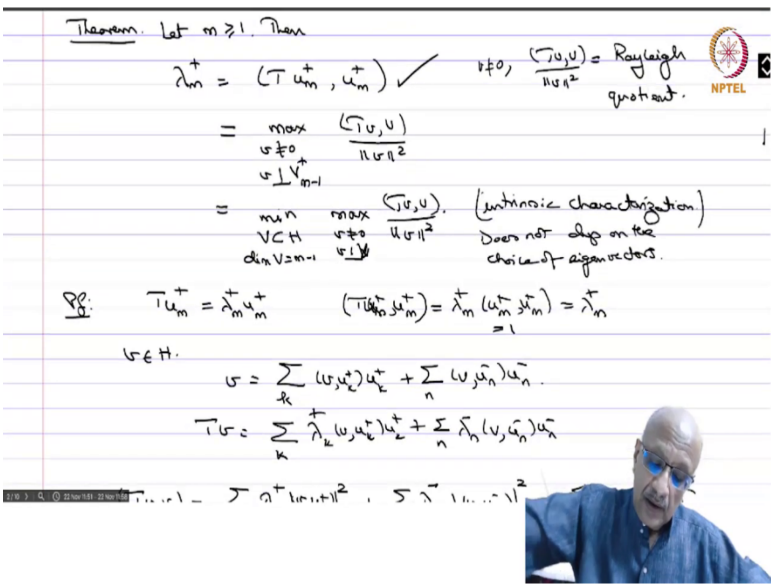
$$\lambda_1^+ \geq \lambda_2^+ \geq \cdots \geq 0 \text{ and } \lambda_1^- \leq \lambda_2^- \leq \cdots \leq 0.$$

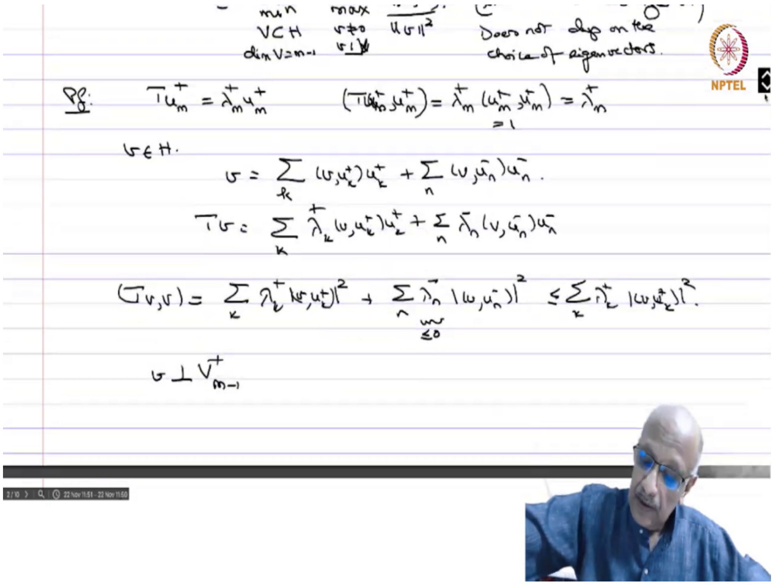
Now, I put less than or equal to or greater than or equal to because they all may not be distinct eigenvalues. So, if we have started with the basis of eigenvectors for each eigenvector, we have chosen an eigenvalue and so, many more than one eigenvector may correspond to the same eigenvalue.

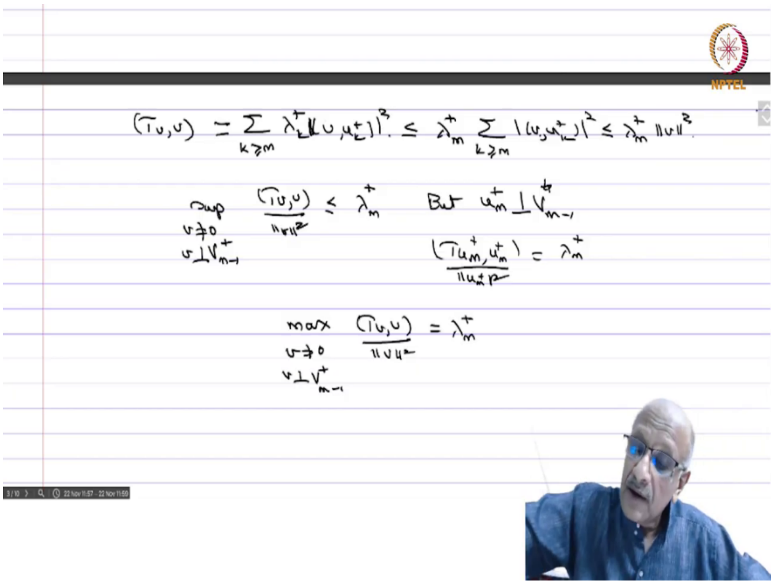
So, for instance there may be an eigenvalue whose geometric multiplicity is 3 that means, there will be 3 eigenvectors in the orthonormal basis which correspond to the same eigenvalue. So, for instance if λ_2, λ_3, λ_4 may all be the same, so like that. So, this numbering by repeating eigenvalues according to their geometric multiplicity. So, if the null space of $\lambda_i - T$, say has dimension 5 then the corresponding λ will occur 5 times consecutively in this list somewhere. And also, all these need not exist all the time, for instance you may have an operator which has only non-negative eigenvalues or only non-positive eigenvalues or a mixture of both, both may be infinite sets one may be finite and so on.

So, all kinds of things are happening we are just giving you a numbering here. So, u_i^+ the eigenvector in the basis corresponding to λ_i^+ and u_i^- is eigenvector in the basis corresponding to λ_i^-. So, these are all the eigenvalues and eigenvectors which we have. So, we have said $V_0 = \{0\}$ and we said $V_m^{\pm}$ to be the span of $\left\{u_1^{\pm}, u_2^{\pm}, \cdot, \cdot, \cdot, u_m^{\pm}\right\}$, that means V_m^+ is the span of $\left\{u_1^+, u_2^+, \cdot, \cdot, \cdot, u_m^+\right\}$ and V_m^- will be the span of $\left\{u_1^-, u_2^-, \cdot, \cdot, \cdot, u_m^-\right\}$. So, this is the notation which we are having.

(Refer Slide Time: 05:00)







So, now we have the following theorem.

Theorem: So, let $m \geq 1$, then $\lambda_m^+ = (Tu_m^+, u_m^+)$. This is also equal to the $\frac{(Tv,v)}{\|v\|^2}$.

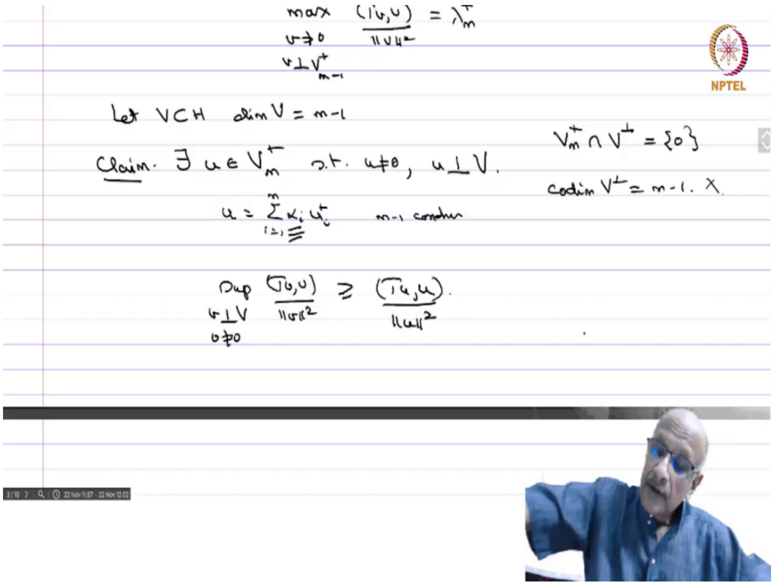
And the third one is the $\frac{(Tv,v)}{\|v\|^2}$.

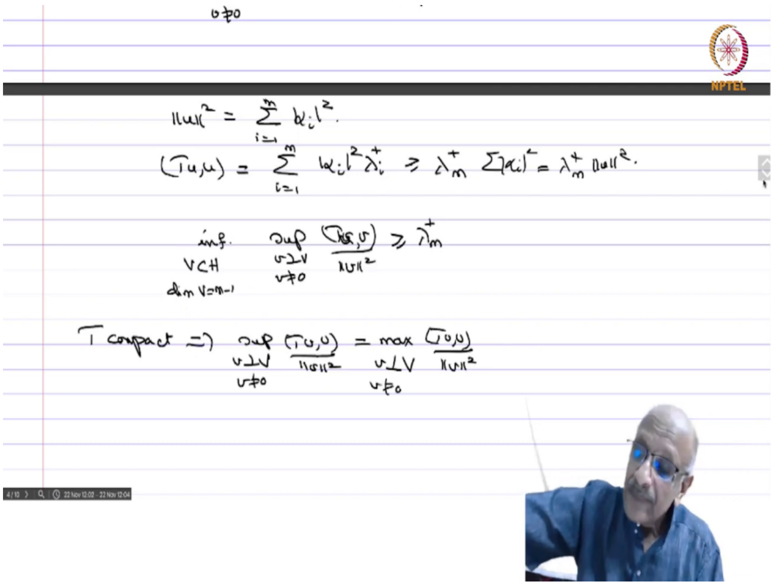
So, let us comment briefly on these things, the first one is just saying u is more or less obvious. So, we will see it in a moment. Now, this $\frac{(Tv,v)}{\|v\|^2}$ for $v \neq 0$ this is called the Rayleigh quotient. So, we are expressing the eigenvalues as the maxima or minima of the Rayleigh quotient over some subspaces. So, there is some constrained optimization problem which we are showing, that is why we call these as the variational characterizations. Now, the first two depend on the orthonormal basis which you have chosen. The basis can be chosen in many ways. So, we are choosing something and so, this depends on that. The third one does not depend on any basis, so this is called the intrinsic characterization, it does not depend on the choice of eigenvectors. So, that is why this is called intrinsic, so it is entirely independent of its basis invariant. So, let us try to prove this.

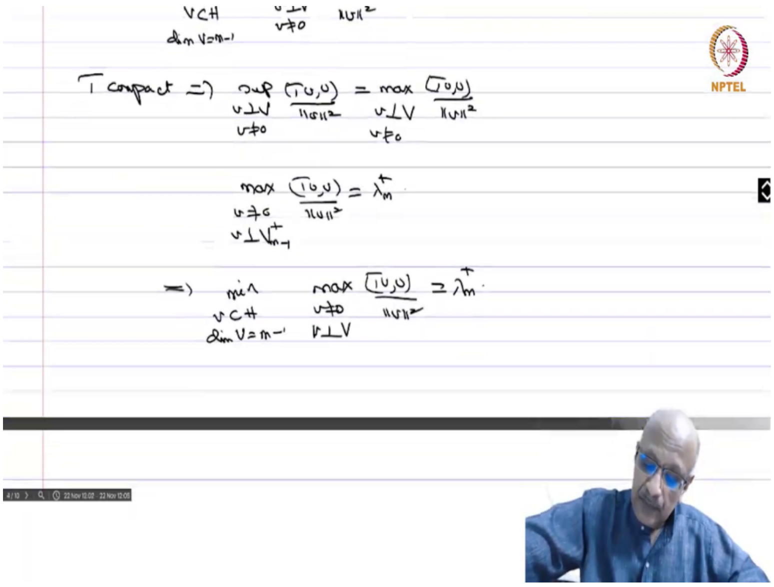
Proof: First one is obvious, so you have $Tu_m^+ = \lambda_m^+ u_m^+ = \lambda_m^+$, so you take the inner product, so $\left(Tu_m^+, u_m^+\right) = \lambda_m^+ (u_m^+, u_m^+)$, and $\left(u_m^+, u_m^+\right) = 1$, and therefore, $\left(Tu_m^+, u_m^+\right) = \lambda_m^+$. So, that proves the first one that is almost immediate. Now, let us see, so let $v \in H$. So, you have an orthonormal basis of eigenvectors and therefore, you can write $v = \sum_k (v, u_k^+) u_k^+ + \sum_n (v, u_n^-) u_n^-$. I am not writing the limits, because as I said, they may be finite, infinite, they may not exist, etc. So, we are just writing sigma over k, sigma over n. And so, what is Tv? $Tv = \sum_k \lambda_k^+ \left(v, u_k^+\right) u_k^+ + \sum_n \lambda_n^- (v, u_n^-) u_n^-$. So, then you take (Tv, v), (Tv, v) is what? So, when you take the inner product (Tv, v), so all cross terms are going to disappear because of the orthogonality only the squared terms are going to remain. So, $(Tv, v) = \sum_k \lambda_k^+ \left|\left(v, u_k^+\right)\right|^2 + \sum_n \lambda_n^- \left|\left(v, u_n^-\right)\right|^2$. And recall that all λ_n's are negative and therefore, $\sum_k \lambda_k^+ \left|\left(v, u_k^+\right)\right|^2 + \sum_n \lambda_n^- \left|\left(v, u_n^-\right)\right|^2 \leq \sum_k \lambda_k^+ \left|\left(v, u_k^+\right)\right|^2$. So, now we have, $v \perp V_{m-1}^-$. So, that means, all the basis up to $m - 1$ all these elements will disappear and therefore, you have $(Tv, v) \leq \sum_{k \geq m} \lambda_k^+ \left|\left(v, u_k^+\right)\right|^2$. So, now, we have numbered the thing positive eigenvalues in the decreasing order, so the biggest of them will be λ_m^+ and then all the highest for the other k it will be less. So, we have $\sum_{k \geq m} \lambda_k^+ \left|\left(v, u_k^+\right)\right|^2$ is less than or equal to the biggest of them. So, that is equal to $\lambda_m^+ \sum_{k \geq m} \left|\left(v, u_k^+\right)\right|^2$ and $\lambda_m^+ \sum_{k \geq m} \left|\left(v, u_k^+\right)\right|^2 \leq \lambda_m^+ \|v\|^2$. Therefore, you have the $\frac{(Tv,v)}{\|v\|^2} \leq \lambda_m^+$ and this is true for all the v's and therefore, in particular $\frac{(Tv,v)}{\|v\|^2} \leq \lambda_m^+$. But

$u_m^+ \perp V_{m-1}$ and therefore, and we have the $\frac{(Tu_m^+, u_m^+)}{\|u_m^+\|^2} = \lambda_m^+$. So, and therefore, that is equal to λ_m^+. So, there is an element here for which this is attained and therefore, you have $\frac{(Tv,v)}{\|v\|^2} = \lambda_m^+$. So, that is the second statement which we have.

(Refer Slide Time: 13:22)





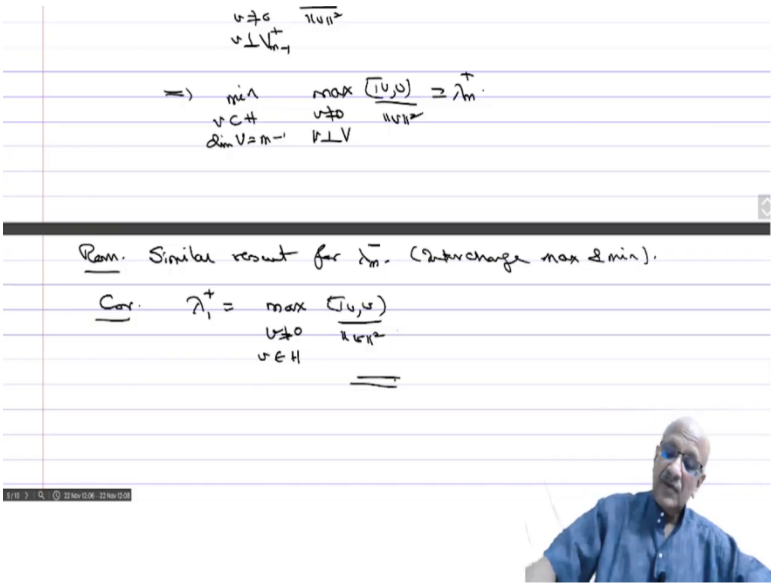


So, now for the third one. So, let V contained in H, $\dim dim\, V = m - 1$, then

Claim: there exists a $u \in V_m^+$ such that $u \neq 0$ and $u \perp V$.

Why is this so? So, we can argue in two different ways. Suppose this is not true, that means there is no non-zero element which is orthogonal to all it. That means $V_m^+ \cap V^\perp = \{0\}$ that is what we are seeing. But $V^\perp = m - 1$ because V is an $m - 1$ dimensional space and therefore, you cannot have an m dimensional space completely non intersecting with it because the codimension is only $m - 1$ and we have m dimension. Therefore, you have definitely, this is a contradiction. Or the other way if you want to look at, is that u is in m dimensional space, so u can be $\sum_{i=1}^{m} \alpha_i u_i^+$. So, you need m constants α_i is to be determined. Now, we are giving by saying it is orthogonal to $m - 1$ dimensional space we are giving $m - 1$ condition and therefore, there is still one more condition free for you and therefore, that will help you to determine the vector u. So, there will be more than one vector in fact and therefore, you always have a u which is non-zero such that u is orthogonal to V. Therefore, $\frac{(Tv,v)}{\|v\|^2} \geq \frac{(Tu,u)}{\|u\|^2}$. So, let us compute $\frac{(Tu,u)}{\|u\|^2}$. So, $u = \sum_{i=1}^{m} \alpha_i u_i^+$ and therefore, $\|u\|^2 = \sum_{i=1}^{m} |\alpha_i|^2$. And (Tu, u) is nothing but again similar to the calculation which we have already done is $\sum_{i=1}^{m} |\alpha_i|^2 \lambda_i^+$. Now, we have again remembered that we are determining the eigenvalues, numbering the eigenvalues in decreasing order and therefore, λ_1^+ will be the biggest λ_m^+ is the last, so $\sum_{i=1}^{m} |\alpha_i|^2 \lambda_i^+ \geq \lambda_m^+ \sum_{i=1}^{m} |\alpha_i|^2$ which is $\lambda_m^+ \|u\|^2$. Therefore, $\frac{(Tu,u)}{\|u\|^2} \geq \lambda_m^+$. Therefore, $\frac{(Tv,v)}{\|v\|^2} \geq \lambda_m^+$ and this is true for any $m - 1$ dimensional space. Therefore, if you take the inf of $V \subset H$, $V = m - 1$, this is still true. Now, T compact implies that $\frac{(Tv,v)}{\|v\|^2}$ is in fact attained, is in fact equal to $\frac{(Tv,v)}{\|v\|^2}$. Because you take a maximizing sequence then (Tv, v) will converge strongly, v will converge weekly and therefore, actually the limit will be maximiser and therefore, it will attain and it will also be orthogonal to V and therefore, it will attain the maximum, so the sup is actually a max. And we have already seen that $\frac{(Tv,v)}{\|v\|^2} = \lambda_m^+$, V_{m-1} is $m - 1$ dimensional space. So, there is one space for which this is attained and therefore, this implies that $\frac{(Tv,v)}{\|v\|^2} = \lambda_m^+$, so that is the third statement which we have proved. So, we have proved all the three statements and we have shown this.

(Refer Slide Time: 19:55)



So, remark,

Remark: Similar theorem, similar result for λ_m^-. So, interchange max and min, you will get the corresponding results.

Now corollary of the previous result. What is λ_1^+? So, λ_1^+ is the $\cdot$. So, but that is $m = 1$, so we have 0 and consequently you will have that it is 0 orthogonal is nothing but the whole space.

Corollary: So, $\lambda_1^+ = \frac{(Tv,v)}{||v||^2}$.

You can also see directly from this, from this statement here. So, you have $(Tv, v) \leq \sum_k \lambda_k^+ |\left(v, u_k^+\right)|^2$ and therefore, this is less than or equal to $\lambda_1^+ \sum_k |\left(v, u_k^+\right)|^2$ because that is the biggest of them all and consequently $\lambda_1^+ ||v||^2$ and then for the u_1^+ you already have attained it and therefore, the maximum over the whole space is in fact λ_1^+ and this is a very useful thing because you get directly a maximization problem which you want to solve for the first eigenvalue and therefore you can find it. So, we will wind up with this and then we will do the exercises.