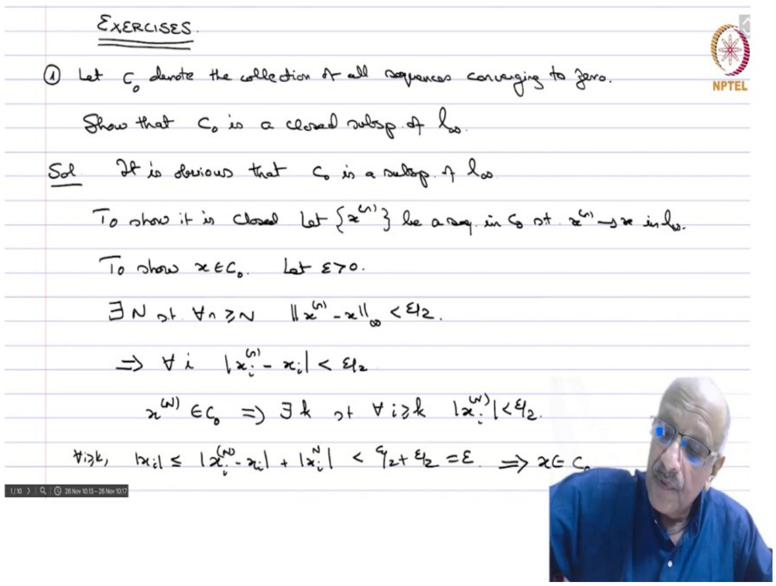
Functional Analysis
Professor S. Kesavan
Department of Mathematics
Institute of Mathematics Science
Lecture 2.1
Exercises

(Refer Slide Time: 00:16)



Exercise 1. Let C_0 denote the collection of all sequences (real or complex) converging to zero, show that C_0 is a closed subspace of l_∞.

Solution: So C_0 is set of all sequences real sequences, if you are working with the real field, complex sequences if you are working with the complex field. Now, it is certainly a subspace of l_∞ because if you have two sequences converging to 0 there is sum also converges to 0. If you have a sequence converging to 0 and you multiply every term by α that will also still converge to 0 and every sequence which is convergent is automatically bounded so it is part of l_∞, so it is clear that C_0 is subspace of l_∞.

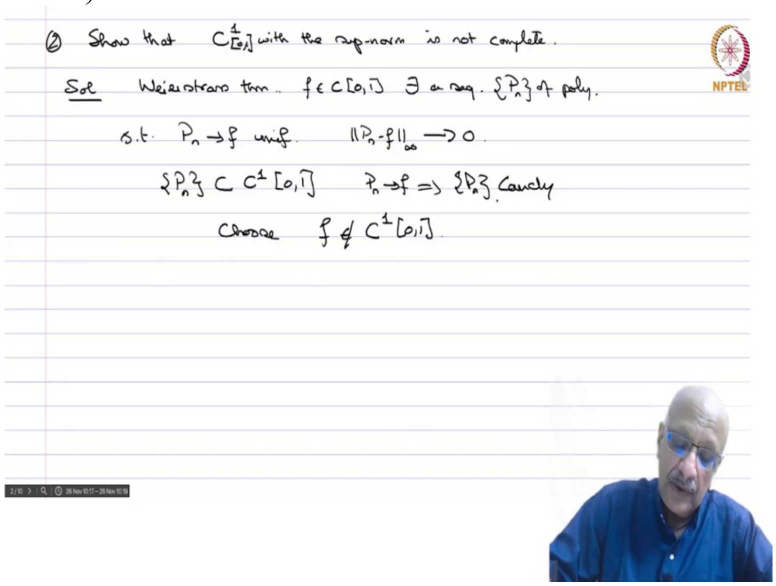
So, we only have to show that it is closed. To show that it is closed, let $(x^{(n)})$ be a sequence in C_0 such that $x^{(n)} \to x$ in l_∞. What must you show to show that $x \in C_0$? Let $\epsilon > 0$. Then there exists N such that for all $n \geq N$, we have $\|x^{(n)} - x\|_\infty < \frac{\epsilon}{2}$. What does this imply? For every i you have mod

$x_i^{(n)} - x_i < \frac{\epsilon}{2}$. also $x^{(N)} \in C_0$. Therefore, there exists a k such that for all $i \geq k$, we have $\left|x_i^{(N)}\right| < \frac{\epsilon}{2}$ (it is a sequence which goes to 0, therefore, after some stage it can be made as small as you like). Therefore, for all $i \geq k$, you have

$$\left|x_i\right| \leq \left|x_i^{(N)} - x_i\right| + \left|x_i^{(N)}\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

Therefore, for all $i \geq k$, $\left|x_i\right| \leq \epsilon$.Tthis means that $x \in C_0$. So, this completes the proof.

(Refer Slide Time: 04:32)



Exercise 2. Show that $C^1[0,1]$ with the sup-norm is not complete.

Solution: This is just very easy. You have sup-norm is the norm. We have Weierstras theorem. Weierstras theorem theorem says if $f \in C^1[0,1]$, then there exists a sequence (P_n) of polynomials, such that $P_n \to f$ uniformly that means $\left\|P_n - f\right\|_\infty \to 0$.

This sequence (P_n) of polynomials are contained in $C^1[0,1]$ because they are all polynomials, hence infinitely differentiable. Now, $P_n \to f$ implies (P_n) is Cauchy. So, But f not necessarily in $C^1[0,1]$. So, choose $f \notin C^1[0,1]$, so it is a continuous function but it is not differentiable, then also you have a a sequence (P_n) of polynomials, such that $P_n \to f$ uniformly. So, this sequence will be a Cauchy sequence which converges outside and therefore this space is not complete.

(Refer Slide Time: 06:38)

③ Let $V = C^1[0,1]$. Define for $f \in C^1[0,1]$

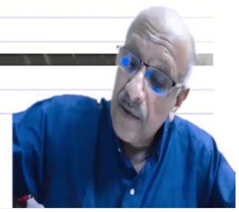
$$\|f\|_{(1)} = \max\{\|f\|_\infty, \|f'\|_\infty\}.$$

Then $\|\cdot\|_{(1)}$ defines a norm on $C^1[0,1]$ and this space is complete.

<u>Sol.</u> $\|f\|_{(1)} \geq 0$, $\|f\|_{(1)} = 0 \Rightarrow \|f\|_\infty = 0 \Rightarrow f = 0$.

$f = 0 \Rightarrow \|f\|_{(1)} = 0$.

$\|\alpha f\|_{(1)} = |\alpha| \|f\|_{(1)}$ obvious.



$\|\alpha f\|_{(1)} = |\alpha| \|f\|_{(1)}$ obvious.

$\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq \|f\|_{(1)} + \|g\|_{(1)}$

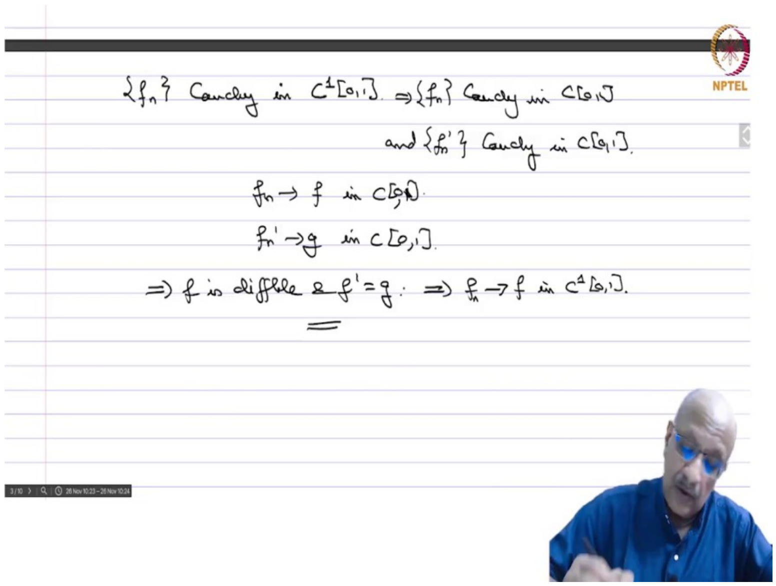
$\|(f+g)'\|_\infty \leq \|f'\|_\infty + \|g'\|_\infty \leq \|f\|_{(1)} + \|g\|_{(1)}$.

$\Rightarrow \|f+g\|_{(1)} \leq \|f\|_{(1)} + \|g\|_{(1)}$

$\{f_n\}$ Cauchy in $C^1[0,1] \Rightarrow \{f_n\}$ Cauchy in $C[0,1]$ and $\{f_n'\}$ Cauchy in $C[0,1]$.

$f_n \to f$ in $C[0,1]$.

$f_n' \to g$ in C



Exercise 3. Now we want to make $C^1[0,1]$ complete. Let $V = C^1[0,1]$ and define

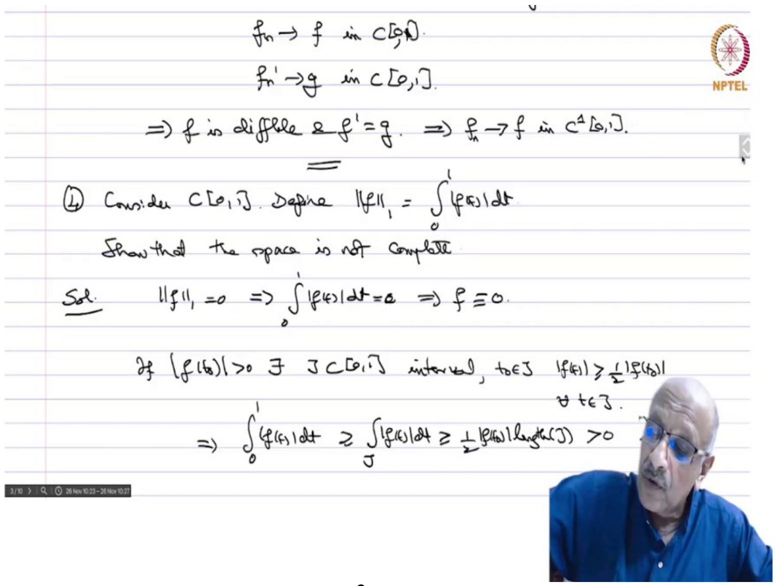
$$\|f\|_1 = max\{\|f\|_\infty, \|f'\|_\infty\}.$$

Then $\|f\|_1$ defines norm on $C^1[0,1]$ and this space, $\|f\|_1 \geq 0$ and norm $\|f\|_1 = 0$ means $\|f\|_\infty = 0$ and therefore, in particular, this implies that $f = 0$. Conversely $f = 0$ implies $\|f\|_1 = 0$ and $\|\alpha f\|_1 = |\alpha| \|f\|_1$ is obvious. What about triangle inequality? So, if you have $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty \leq \|f\|_1 + \|g\|_1$ and $\|(f+g)'\|_\infty \leq \|f'\|_\infty + \|g\|_\infty \leq \|f\|_1 + \|g\|_1$. Therefore you have the maximum is also true, this means that $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.

Now, what about the completeness? If (f_n) is Cauchy in $C^1[0,1]$ then (f_n) is Cauchy in $C[0,1]$ and (f_n') is Cauchy in $C[0,1]$.

So, let us take $f_n \to f$ in $C[0,1]$, and $f_n' \to g$ in $C[0,1]$, then what does this mean? We now know convergence in ¿∨.∨$|_\infty$ is the same as uniform convergence. So, you have $f_n \to f$, and $f_n' \to g$ uniformly. Then from real analysis we know that that f is differentiable and $f' = g$ and therefore this implies that $f_n \to f$ in $C^1[0,1]$ and therefore $C^1[0,1]$ is complete. (We are using the theorem in analysis, which says that if the derivative converges uniformly and in fact you need much less, if the function converges at one point and the derivative converges uniformly, then we say that the function itself converges uniformly to a differentiable function whose derivative is the limit).

(Refer Slide Time: 11:42)



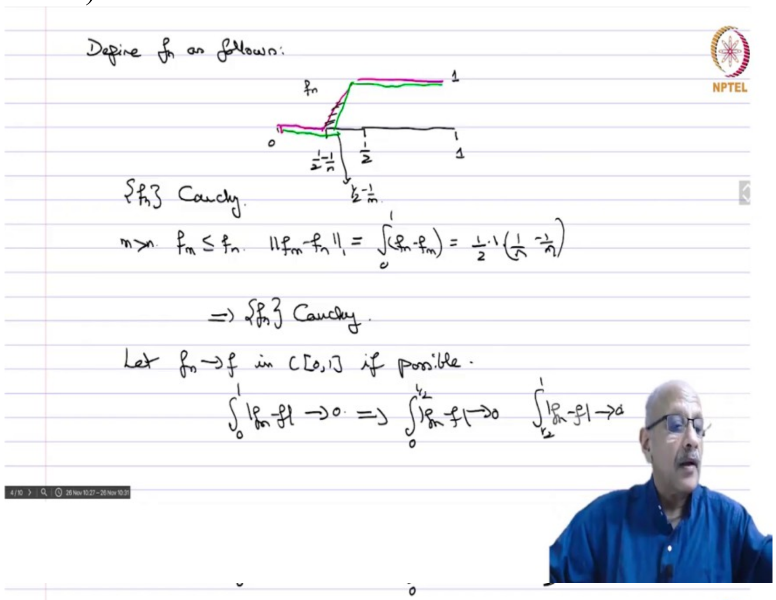
Exercise 4. Consider $C[0,1]$ define $\|f\|_1 = \int_{[0,1]} |f(t)|\,dt$. Show that the space is not complete.

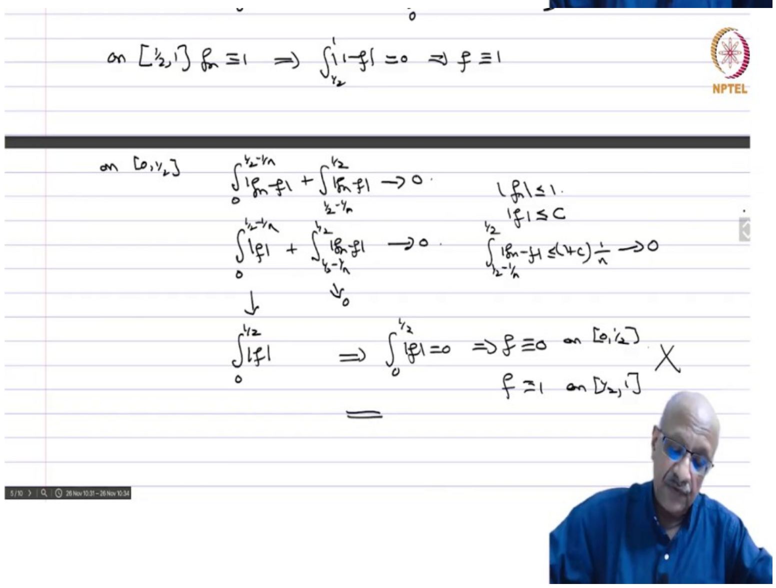
Solution: $\|f\|_1$ is non negative and $\|\alpha f\|_1 = |\alpha|\|f\|_1$. Triangle inequality is also trivial. Also, if you have zero function, the integral is 0, so, $\|f\|_1 = 0$. Conversely, if $\|f\|_1 = 0$ i.e., $\int_{[0,1]} |f(t)|\,dt = 0$. This implies that $f = 0$. Why? Because if f were not 0 then there exists t_0 such that $|f(t_0)| > 0$. Hence, there exists $J \subseteq [0,1]$ with $t_0 \in J$ and $|f(t)| \geq \frac{1}{2}|f(t_0)|, \forall\, t \in J$.

This will imply that $\int_{[0,1]} |f(t)|\,dt \geq \int_J |f(t)|\,dt \geq \frac{1}{2}|f(t_0)|\, length(J) > 0$ that is a contradiction. So, we have that if $\|f\|_1 = 0$ then $f = 0$.

So, now we have to show that it is not complete.

(Refer Slide Time: 13:58)





Define (f_n) in the following fashion.

$f_n = 0$ on $[0, \frac{1}{2} - \frac{1}{n}]$ then f_n is linear on $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}]$ and $f_n = 1$ on $[\frac{1}{2}, 1]$.

Now we want to show that the function (f_n) is Cauchy. Let us assume that $m > n$. Then, you have that $f_m \leq f_n$ and therefore $\|f_m - f_n\|_1 = \int_{[0,1]} |(f_m - f_n)(t)| dt = \frac{1}{2}(\frac{1}{n} - \frac{1}{m})$. And this implies that (f_n) is Cauchy. So, now let us see whether it converges at all.

Let $f_n \to f$ in $C[0,1]$. That means $\int_{[0,1]} |(f_n - f)(t)| dt \to 0$. And that implies $\int_{[0,\frac{1}{2}]} |(f_n - f)(t)| dt \to 0$ and $\int_{[\frac{1}{2},1]} |(f_n - f)(t)| dt$. But $[\frac{1}{2},1]$, $f_n = 1$, therefore, $\int_{[\frac{1}{2},1]} |(f_n - f)(t)| dt = \int_{[\frac{1}{2},1]} |1 - f(t)| dt = 0$. This implies $f = 1$ on $[\frac{1}{2},1]$.

So, on $\left[0,\frac{1}{2}\right]$, we have

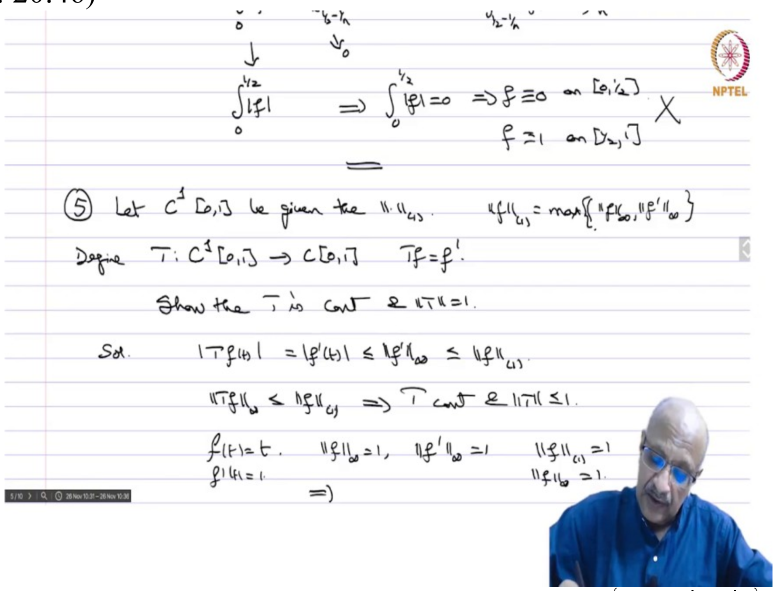
$$\int_{[0,\frac{1}{2}-\frac{1}{n}]} |(f_n - f)(t)| dt + \int_{[\frac{1}{2}-\frac{1}{n},\frac{1}{2}]} |(f_n - f)(t)| dt \to 0$$

$$\text{i.e., } \int_{[0,\frac{1}{2}-\frac{1}{n}]} |f(t)| dt + \int_{[\frac{1}{2}-\frac{1}{n},\frac{1}{2}]} |(f_n - f)(t)| dt \to 0$$

Now, f_n are all bounded and they are all less than equal to 1. Further, since $(f¿¿n)¿$converges to f, $|f| \leq C$ (it is a continuous function on a compact set, therefore, it has to be less than equal to constant). Thus, $\int_{[\frac{1}{2}-\frac{1}{n},\frac{1}{2}]} |(f_n - f)(t)| dt \leq (1 + C)\, length\left(\left[\frac{1}{2} - \frac{1}{n}, \frac{1}{2}\right]\right) = \frac{(1+C)}{n} \to 0.$

Also, by the Monotone convergence theorem $\int_{[0,\frac{1}{2}-\frac{1}{n}]} |f(t)| dt \to \int_{[0,\frac{1}{2}]} |f(t)| dt$. Therefore, $\int_{[0,\frac{1}{2}]} |f(t)| dt = 0.$ This means $f = 0$ on $[0,\frac{1}{2}]$ and we know that $f = 1$ on $[\frac{1}{2},1]$ and that is a contradiction because you have a continuous function and it is not possible. Therefore, (f_n) is Cauchy, but it will not converge to anything.

(Refer Slide Time: 20:46)



Exercise 5. Let us consider $C^1[0,1]$ with the $\|.\|_1 \, i.e.,\ \|f\|_1 = max\{\|f\|_\infty, \|f'\|_\infty\}$.

Define $T: C^1[0,1] \mapsto C[0,1]$ as $T(f) = f^{'}$. Show that T is continuous and in fact, $\|T\| = 1$.

<u>*Solution.:*</u> $¿T(f)(t)\vee¿|f^{'}(t)| \le \|f^{'}(t)\|_\infty \le \|f\|_1$. Therefore, $\|T(f)\|_\infty \le \|f\|_1$. This implies that T is continuous and $\|T\| \le 1$.

Now, you take $f(t) = t$ then $\|f\|_\infty = 1 = \|f\,'\|_\infty$ and therefore the maximum is realized and therefore you have $\|T\| = 1$.