Functional Analysis
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Lecture No. 69
Spectrum of a compact self-adjoint operator

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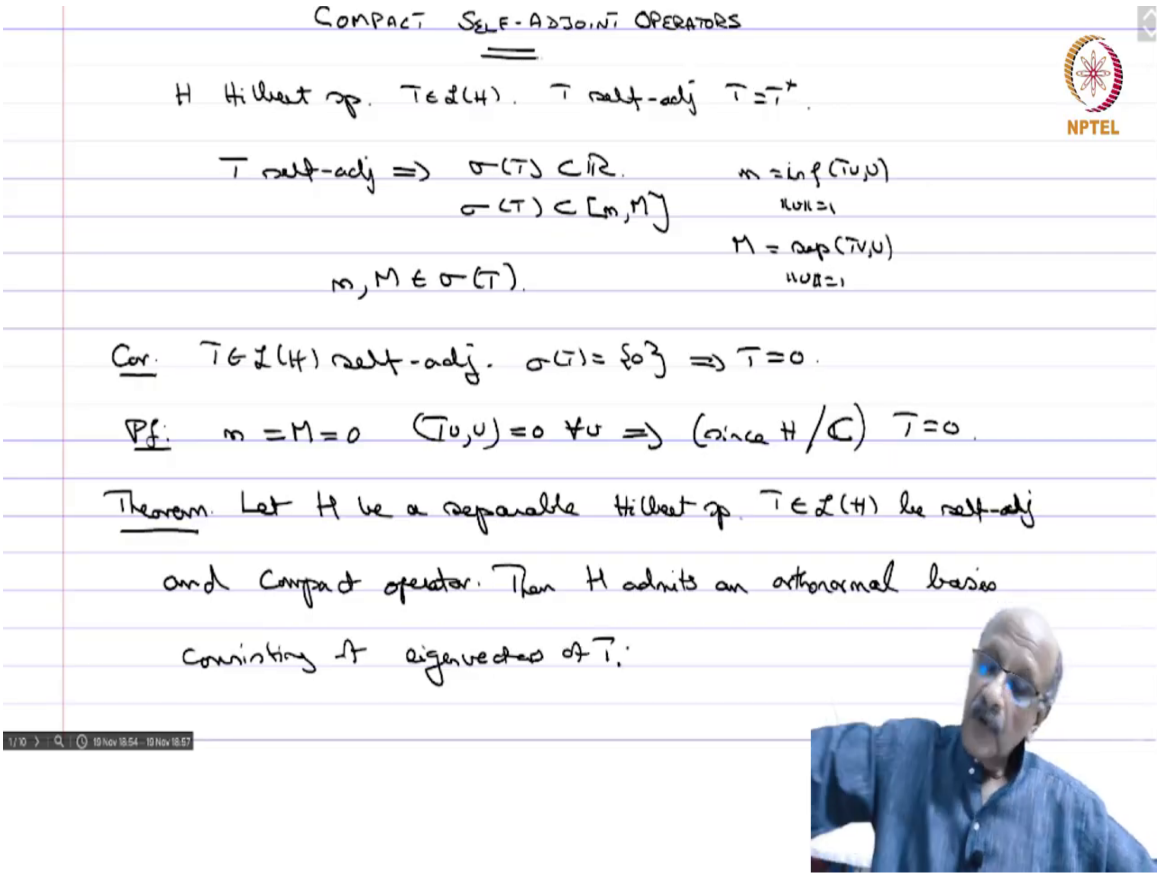

We will now study compact self-adjoint operators.

So, H Hilbert space and $T \in L(H)$. We say T is self-adjoint, if $T = T^*$ and we have already seen the following thing.

T is self-adjoint $\Rightarrow$ $\sigma(T) \subset \mathbb{R}$. And in fact, we have also seen that $\sigma(T) \subset [m, M]$, where $m = \inf_{||v||=1}(Tv, v)$ and capital $M = \sup_{||v||=1}(Tv, v)$. And you also saw that $m, M \in \sigma(T)$. This was done in the exercises of the Hilbert space chapter.

Corollary: $T \in L(H)$ self-adjoint and $\sigma(T) = \{0\} \Rightarrow T = 0$.

Proof. $\sigma(T) = \{0\}$ means $m = M = 0$. Therefore, you have $(Tv,\, v) = 0\ \forall v$ and this implies $T = 0$, since H is over the complex numbers, we saw that it is not true for the reals.

Theorem: Let H be a separable Hilbert space and $T \in L(H)$ be self-adjoint and compact operator. Then, H admits an orthonormal basis consisting of eigenvectors of T. It is obviously countable because H is a separable Hilbert space.

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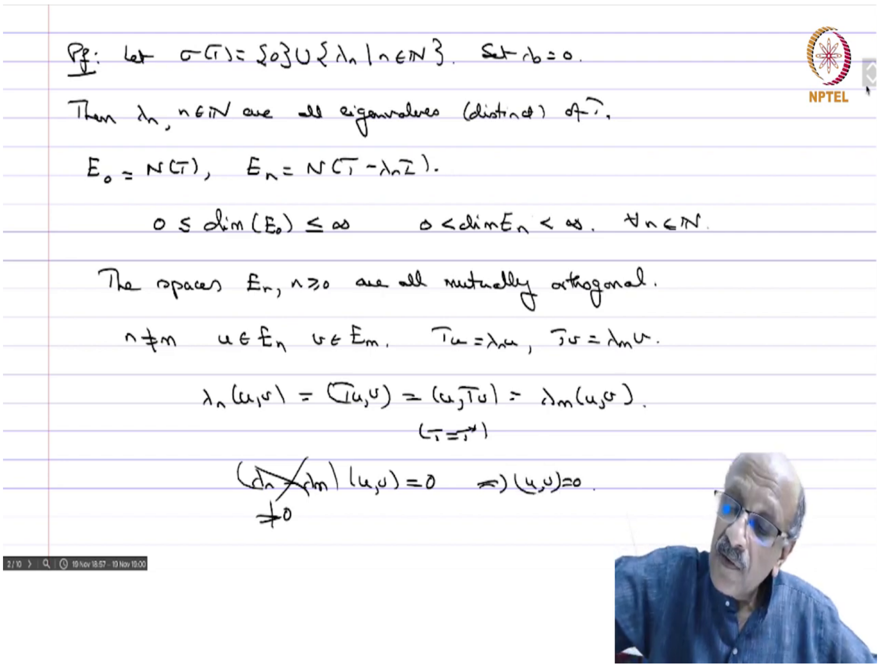

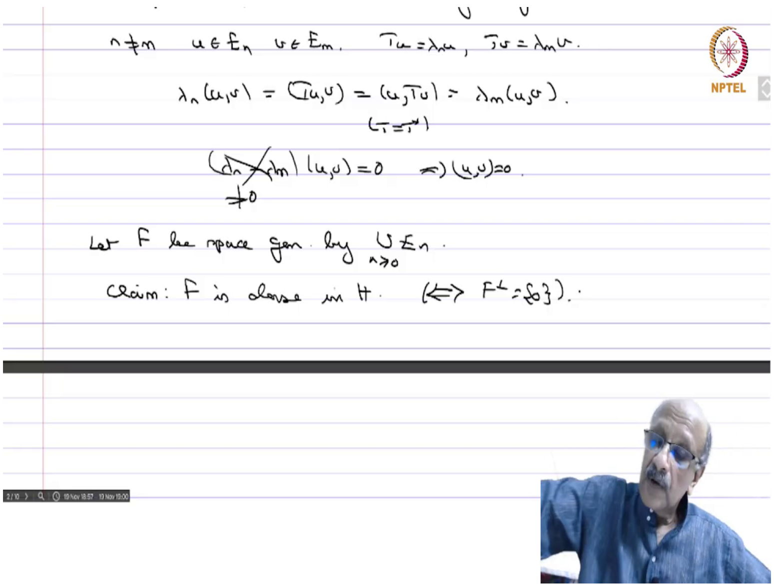

Proof. Let $\sigma(t) = \{0\} \cup \{\lambda_n \,/\, n \in \mathbb{N}\}$. Set $\lambda_0 = 0$ and then λ_n, $n \in \mathbb{N}$ are all distinct eigenvalues of T. So, we say $E_0 = N(T)$ and $E_n = N(T - \lambda I)$. Then what do you know about E_0? E_0 could be singleton 0. So, $0 \leq \dim(E_0) \leq \infty$. On the other hand, $0 < \dim(E_n) < \infty$, $\forall n \in \mathbb{N}$. So, the space E_n, $n \geq 0$ are all mutually orthogonal? Let $n \neq m$. So, you take $u \in E_n$, $v \in E_m$. So, $Tu = \lambda_n u$ and $Tv = \lambda_m v$. So, then if you take $\lambda_n (u, v) = (Tu, v) = (u, Tv) = \lambda_m (u, v)$, because $T = T^*$ and λ_m is real. So, $(\lambda_n - \lambda_m)(u, v) = 0$, and $(\lambda_n - \lambda_m) \neq 0$ and therefore, you have that$(u, v) = 0$. So, all these spaces are mutually orthogonal to each other. Let F be the space generated by $\bigcup_{n \geq 0} E_n$,so you put all these spaces together and you have. Claim F is dense in H, which is equivalent to saying $F^{\perp} = \{0\}$. Because, if you have a vector vanishing on F, then it vanishes everywhere, that is the Hahn-Banach way of showing a space is dense and therefore, showing $F^{\perp} = \{0\}$ is equivalent to showing F is this.

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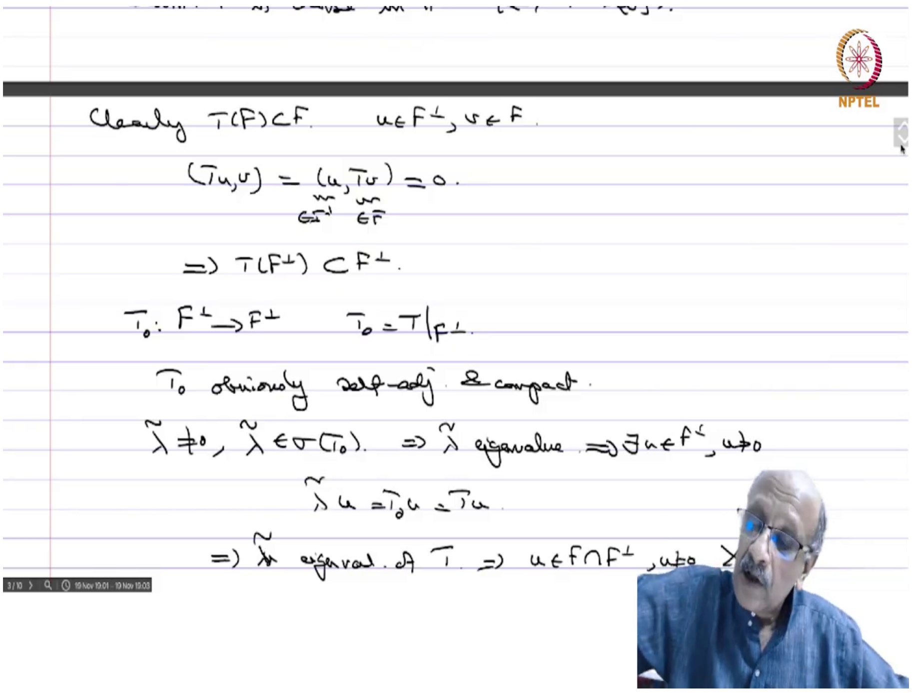

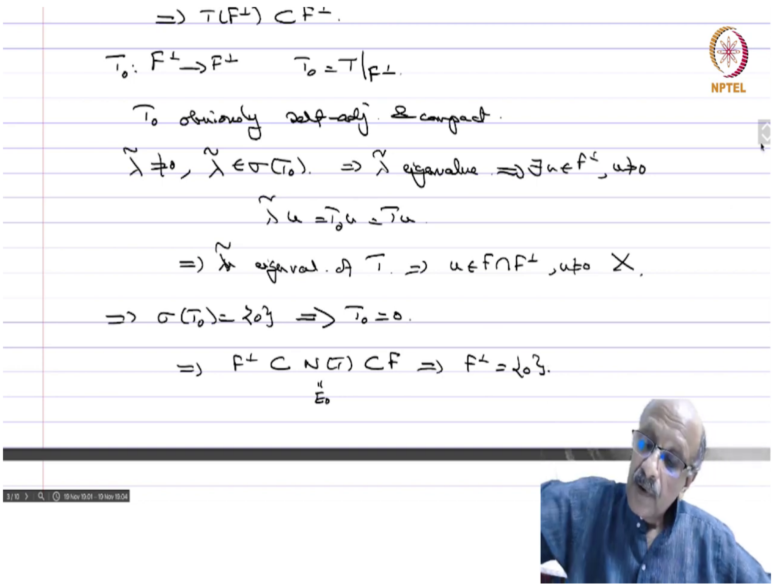

So, clearly $T(F) \subset F$ because any eigenvalue is the same. So, it will be in the same E_n and linear combinations will also be, so $T(F) \subset F$. Now, if $u \in F^{\perp}$ and $v \in F$. You have $(Tu, v) = (u, Tv)$. Now, $v \in F$, so Tv is also in F and $u \in F^{\perp}$ and therefore, $(Tu, v) = (u, Tv) = 0$.

So, this means $T(F^{\perp}) \subset F^{\perp}$. You take $T_0 : F^{\perp} \to F^{\perp}$, $T_0 = T|_{F^{\perp}}$. So, T_0 is obviously self-adjoint and compact. Let $\tilde{\lambda} \neq 0$, $\tilde{\lambda} \in \sigma(T_0)$, so T_0 is self-adjoint compact operator, $F^{\perp}$ is closed. So, it is a Hilbert space on its own and you have a compact self-adjoint operator in it and therefore, it will have a spectrum. So, assume it has a non-zero element in the spectrum, then implies it is an eigenvalue because it is compact, implies that $\exists\, u \in F^{\perp},\ u \neq 0$. So, as I said, $\tilde{\lambda} u = T_0 u$, but $T_0 u = Tu$. This implies that $\tilde{\lambda}$ is an eigenvalue of T, but we have lumped all the eigenvalues together in the space F, that means $u \in F \cap F^{\perp}$ and $u \neq 0$, which is a contradiction, you cannot have that. This implies that $\sigma(T_0) = \{0\}$, and by our earlier corollary we have $T_0 = 0$. This implies that $F^{\perp} \subset N(T) \subset F$ .This this implies that $F^{\perp} = \{0\}$, that is the only way it is possible.

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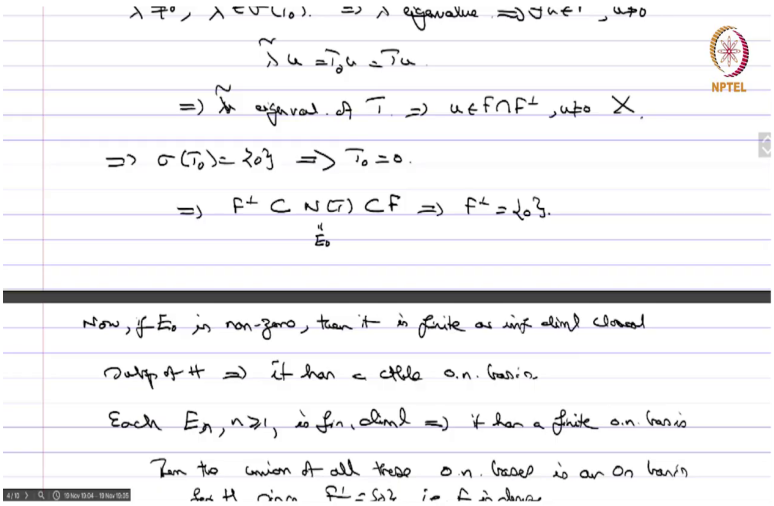

Now, if E_0 is non-zero, then it is a finite or infinite dimensional closed subspace of H and therefore, implies it has a countable orthonormal basis. Each E_n, $n \geq 1$ is finite dimensional implies it has a finite orthonormal basis, then the union of all this orthonormal basis is an orthonormal basis for H, since $F^{\perp} = \{0\}$, F is dense. Because the union is orthonormal is clear, if x is orthogonal to all of these basis vectors, then x is orthogonal to the space F and when we saw that $F^{\perp} = \{0\}$ since F is dense and therefore, it is a thing. So, this proves that you have an orthonormal basis for H. So, next we will look at a variational characterization of the eigenvalues.