Functional Analysis
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Lecture No. 68
Spectrum of a compact operator - Part 2

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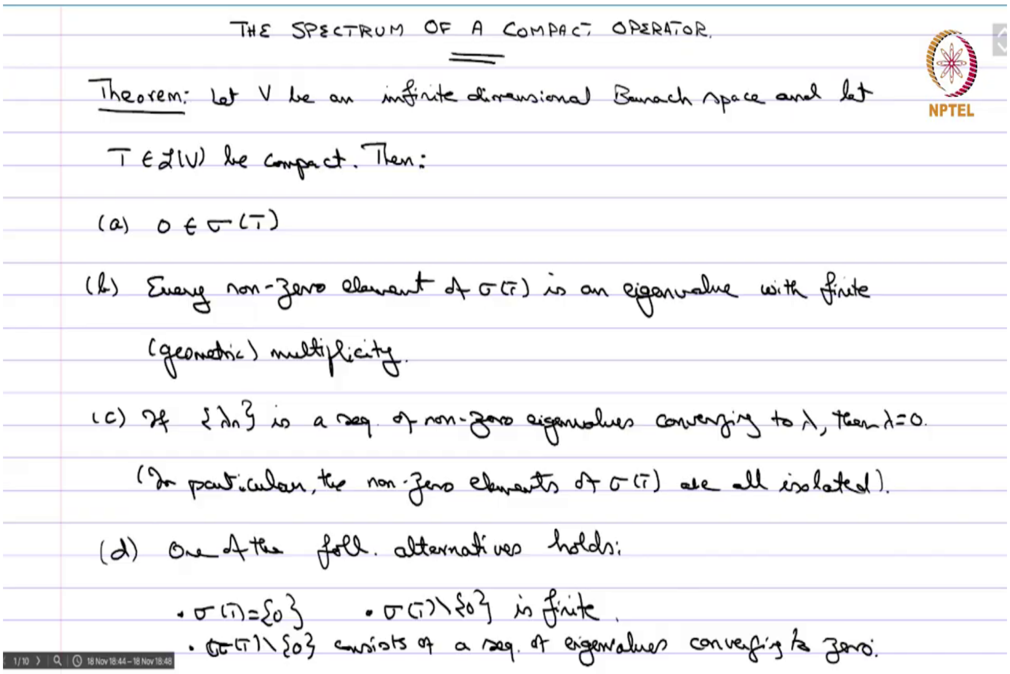

We will now discuss the spectrum of a compact operator. Given a matrix, a spectrum is just the set of all eigenvalues. In the infinite dimensional case, we saw several examples, you could have eigenvalues, you may not have eigenvalue spectrum consisting of various things, all kinds of possibilities occur. Now, we will study the case of a compact operator which is more like the finite dimensional case.

So, more precisely we have the following theorem.

Theorem. Let V be an infinite dimensional Banach space and let $T \in L(V)$ be compact. Then:

(a) $0 \in \sigma(T)$.

(b) Every non-zero element of $\sigma(T)$ is an eigenvalue with finite geometric multiplicity. What is geometric multiplicity? It is nothing but the dimension of the kernel or null space of $T - \lambda I$.

(c) If $\{\lambda_n\}$ is a sequence of non-zero eigenvalues converging to λ, then $\lambda = 0$. So, in particular, the non-zero elements of $\sigma(T)$ (spectrum) are all isolated because nothing can converge to them.

(d) one of the following alternatives holds, you can have

- $\sigma(T) = \{0\}$. There is nothing else in the spectrum,
- $\sigma(T)\backslash\{0\}$ is finite
- $\sigma(T)\backslash\{0\}$ consists of a sequence of eigenvalues converging to 0. So, one of these three will hold.

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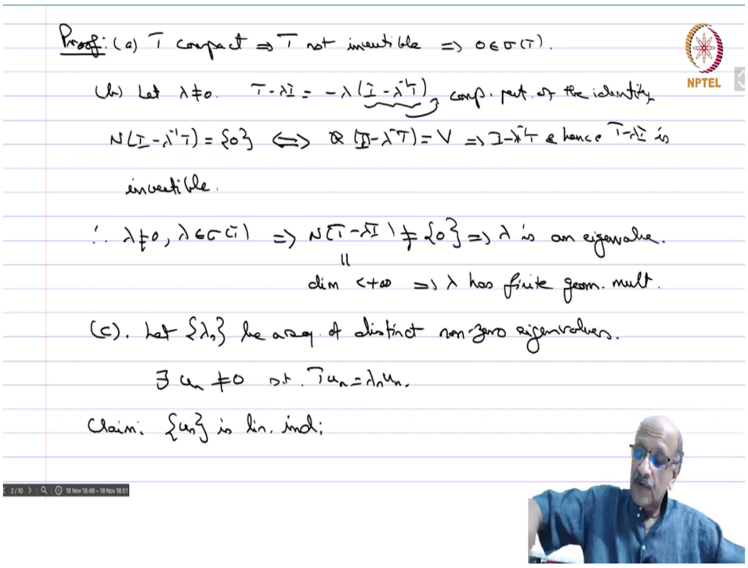

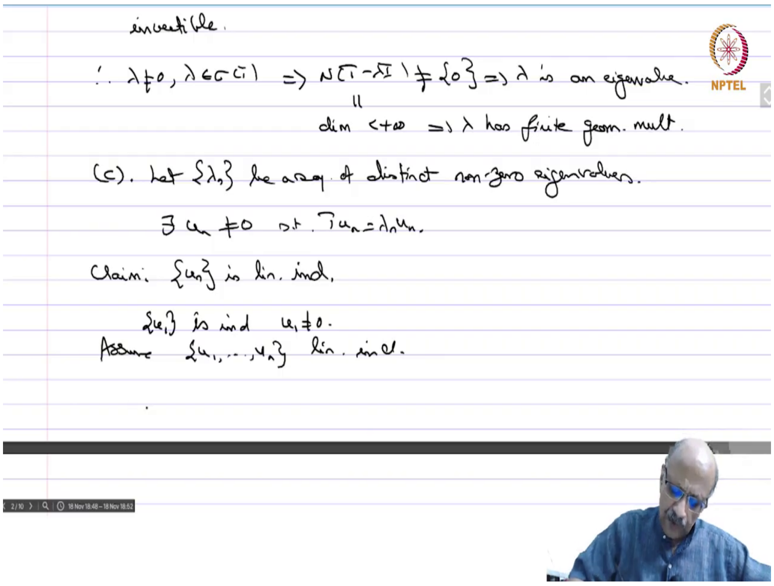

Proof (a). We already saw that composition of compact map is compact, an identity map is not compact in infinite dimensional space because the unit ball is not compact and therefore, you have that compact operator cannot be invertible, so T compact $\Rightarrow$ T not invertible.

And therefore, this we have already seen, therefore this 0 has to belong to $\sigma(T)$.

(b) Let $\lambda \neq 0$ and then you write $T - \lambda I =- \lambda(I - \lambda^{-1}T)$ and this is a compact perturbation of the identity. Therefore, the Riesz-Fredholm theory holds and $N(I - \lambda^{-1}T) = \{0\}$ if and only if $R(I - \lambda^{-1}T) = V$.

So, 1-1 if and only if onto and therefore, $I - \lambda^{-1}T$ is invertible and hence $T - \lambda I$ is invertible. Therefore, $\lambda \neq 0, \lambda \in \sigma(T) \Rightarrow N(T - \lambda I) \neq \{0\}$ and therefore, implies λ is an eigenvalue. And again, by the Riesz-Fredholm theory this dimension is finite and therefore, λ has finite geometric multiples, so that proves (b).

(c) Let $\{\lambda_n\}$ be a sequence of distinct non-zero eigenvalues, then there exists $u_n \neq 0$ such such that $Tu_n = \lambda_n u_n$. So, claim $\{u_n\}$ is linearly independent. The set of all eigenvectors of the distinct eigenvalues form a linearly independent set. $\{u_1\}$ is independent because $u_1 \neq 0$. So, assume $\{u_1, \ldots, u_n\}$ is linearly independent.

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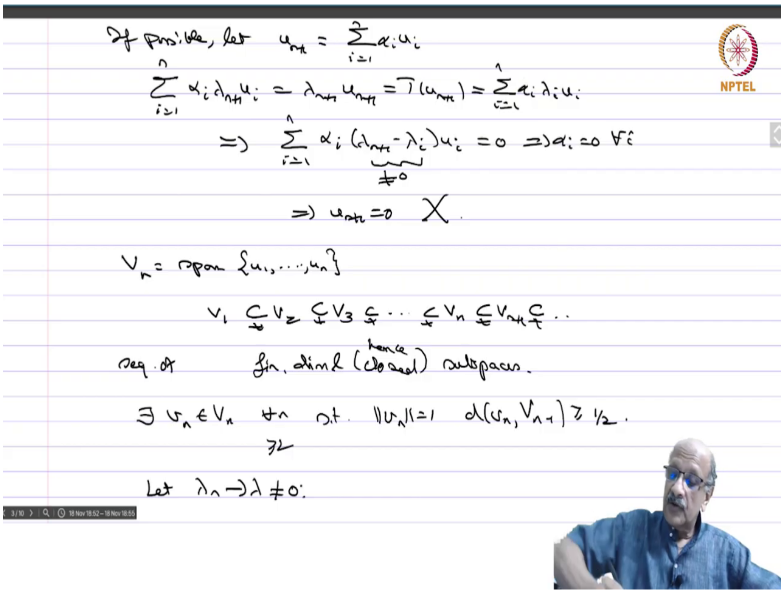

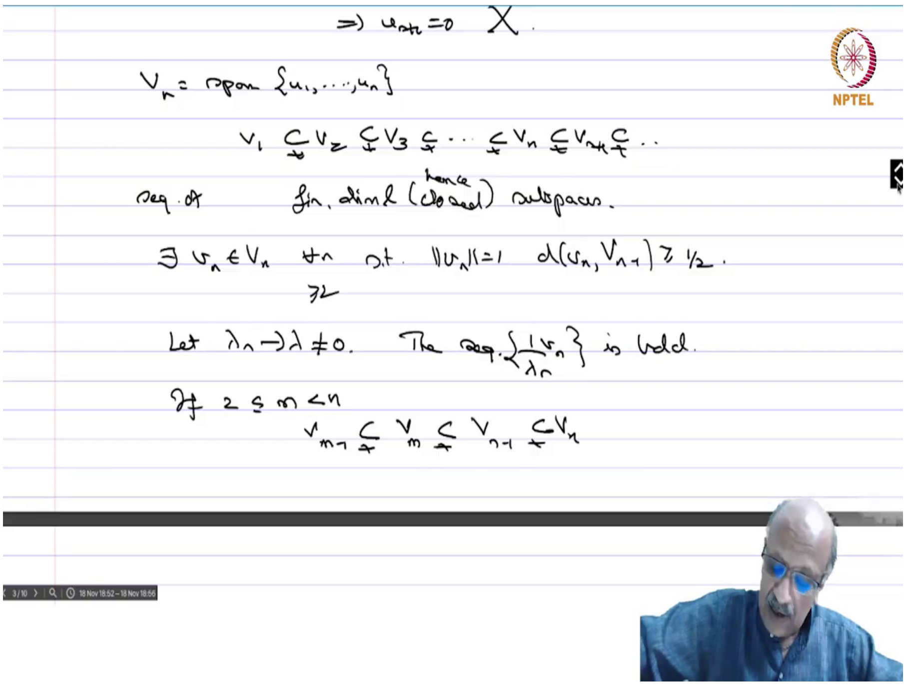

If possible, let $u_{n+1} = \sum_{i=1}^{n} \alpha_i u_i$, that means it is not linearly independent since the first n are linearly independent and you are adding to make it dependent that means the last new one is a linear combination of the previous ones. Now, you multiply both sides by λ_{n+1}. So,

$$\sum_{i=1}^{n} \alpha_i \lambda_{n+1} u_i = \lambda_{n+1} u_{n+1} = T(u_{n+1}) = \sum_{i=1}^{n} \alpha_i T u_i = \sum_{i=1}^{n} \alpha_i \lambda_i u_i$$

$\Rightarrow \sum_{i=1}^{n} \alpha_i (\lambda_{n+1} - \lambda_i) u_i = 0$. But you know $\{u_1, . . ., u_n\}$ is linearly independent and $\lambda_{n+1} - \lambda_i \neq 0$. So, this implies that $\alpha_i = 0, \ \forall i$ and that implies that $u_{n+1} = 0$ and that is a contradiction. You cannot have an eigenvector which is zero. Therefore, the u_n's are linearly independent. Now, you say, $V_n = span \ \{u_1, . . ., u_n\}$. So, this is an n-dimensional subspace therefore closed. And therefore, we have an increasing sequence,

$V_1 \subset V_2 \subset V_3 \subset . . . \subset V_n \subset V_{n+1} \subset \ . . .$ and so on. So, strictly containing a sequence of finite dimensional (hence closed) subspaces. Now we can apply the Riesz lemma you have a closed subspace therefore, there exists a $v_n \in V_n$ such for each $n \geq 2$ such that $||v_n|| = 1$ and $d(v_n, V_{n-1}) \geq 1 - \epsilon$. We will take $\epsilon = \frac{1}{2}$, thus $d(v_n, V_{n-1}) \geq \frac{1}{2}$.

Let $\lambda_n \rightarrow \lambda \neq 0$ if possible, then we will get a contradiction and therefore, $\{\lambda_n\}$ has to only converge to 0. So, look at the sequence, $\{\frac{1}{\lambda_n} v_n\}$ is bounded. Why is it bounded? $||v_n|| = 1$ λ_n converges to $\lambda \neq 0$, so it is bounded away from 0.

So, $\{\frac{1}{\lambda_n}\}$ is a bounded sequence and therefore, $\{\frac{1}{\lambda_n} v_n\}$ is a bounded sequence and also if $2 \leq m < n$, What do you have? $V_{m-1} \subset V_m \subset V_{n-1} \subset V_n$, because $n > m$.

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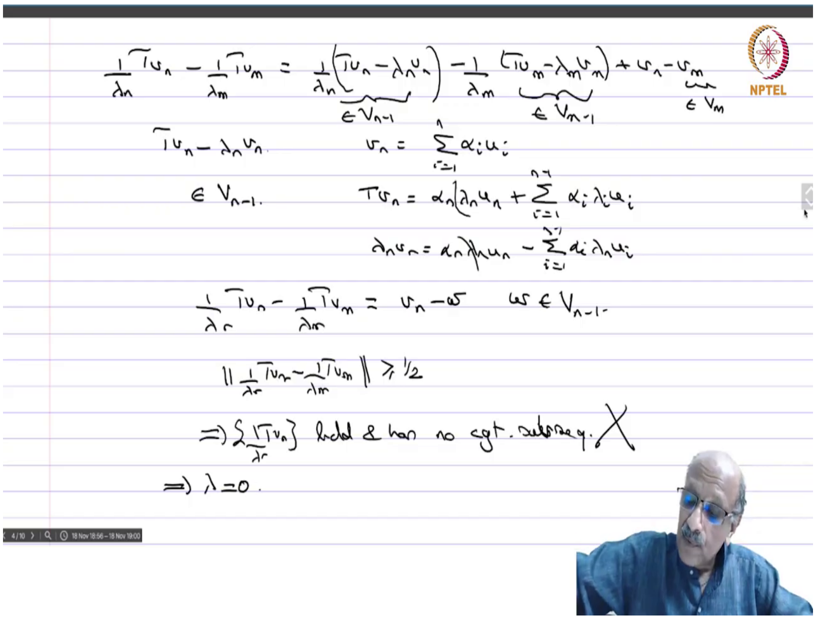

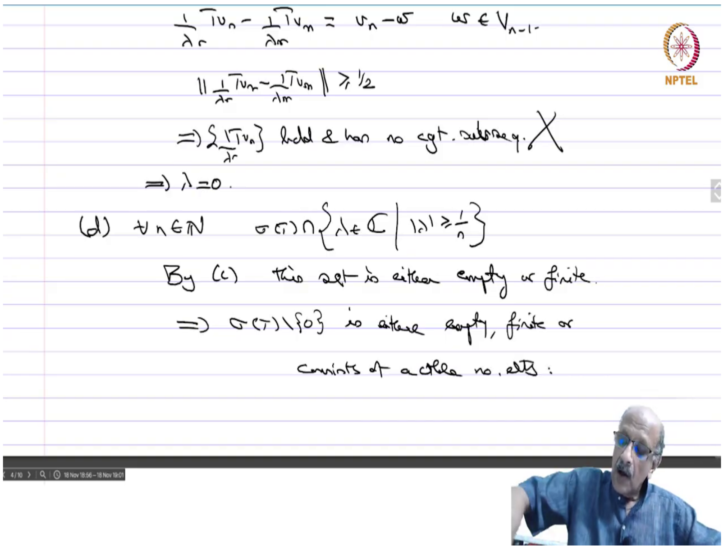

Now, $\frac{1}{\lambda_n} T v_n - \frac{1}{\lambda_m} Tv_m = \frac{1}{\lambda_n}(Tv_n - \lambda_n v_n) - \frac{1}{\lambda_m}(Tv_m - \lambda_m v_m) + (v_n - v_m)$.Now, you can look at $Tv_n - \lambda_n v_n$. What about this? So, $v_n = \sum_{i=1}^{n} \alpha_i u_i \Rightarrow Tv_n = \alpha_n \lambda_n u_n + \sum_{i=1}^{n-1} \alpha_i \lambda_i u_i$ then $\lambda_n v_n = \alpha_n \lambda_n u_n + \sum_{i=1}^{n-1} \alpha_i \lambda_i u_i \Rightarrow Tv_n - \lambda v_n \in V_{n-1}$.

Similarly, $(Tv_m - \lambda_m v_m) \in V_{m-1}$, i.e., $(Tv_m - \lambda_m v_m) \in V_{n-1}$, and $v_m \in V_{n-1}$.

$\frac{1}{\lambda_n} T v_n - \frac{1}{\lambda_m} T v_m = v_n - w$, where $w \in V_{n-1}$. Therefore, $||\frac{1}{\lambda_n} T v_n - \frac{1}{\lambda_m} T v_m|| \geq \frac{1}{2}$ by choice and this is a contradiction because $\{\frac{1}{\lambda_m} T v_m\}$ (T is a compact operator) and therefore, it must have a convergent subsequence. $||\frac{1}{\lambda_n} T v_n - \frac{1}{\lambda_m} T v_m|| \geq \frac{1}{2}$ implies that $\{\frac{1}{\lambda_n} T v_n\}$ is bounded and has no convergent subsequence and that is the contradiction to the compactness. Therefore, this implies that λ has to be equal to 0, so that proves (c) .

(d) $\forall n \in \mathbb{N}$, take $\sigma(T) \cap \{\lambda \in \mathbb{C} / |\lambda| \geq \frac{1}{n}\}$. So, this is a compact closed set, this is compact, so the intersection is compact. So, if it is compact, if it has an infinite number of elements, then there must be a convergent subsequence which should convert to something whose norm is bigger than or equal to $\frac{1}{n}$ that is not possible by part c. So, by (c), this set is either empty or finite and therefore, this implies that $\sigma(T) \backslash \{0\}$ is the union of all these sets is either empty, finite, or consists of a countable number of elements.

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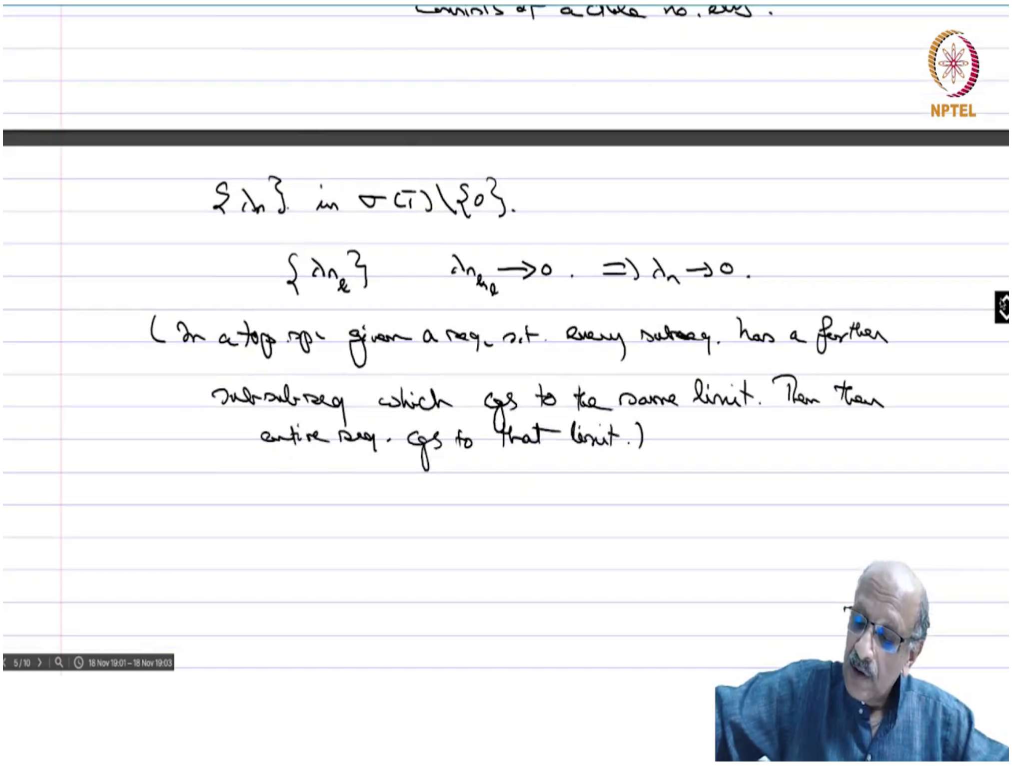

So, it is a countable number of elements, so you can number them as some sequence $\{\lambda_n\}$ in $\sigma(T)\backslash\{0\}$. Then you take any subsequence $\{\lambda_{n_k}\}$. So, that is instantly in the compact set, $\sigma(T)$ and therefore, there exists a further subsequence $\{\lambda_{n_{k_l}}\}$ converge and that we just saw has to converge only to 0.

So, every subsequence has a further subsequence which converges to 0 and this implies that $\lambda_n \rightarrow 0$. So, this is a very beautiful topological fact, very useful, very trivial observation, in a topological space given a sequence such that every subsequence has a further subsequence which converges to the same limit, then the entire sequence converges to that limit.

So, whatever may be the sequence or subsequence, you choose the limit that is always the same, then the entire sequence converges to that limit. Just think about it, it is just a two-minute thing and you can easily show that it is very very useful when you have convergence of a sub sequence and you want to assert something about the convergence of the whole sequence you show that the limit is independent of the sub sequence chosen and therefore, the entire sequence will converge. So, a very very useful topological tool.

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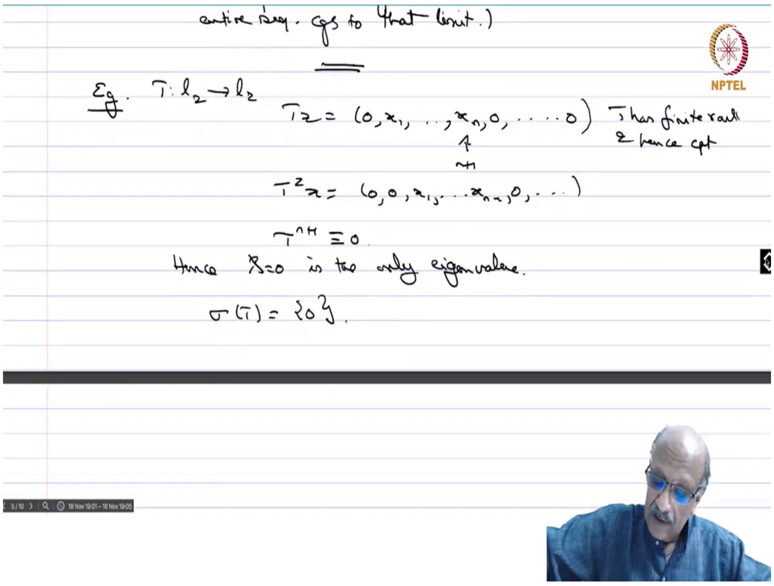

Example. $T\colon\ V \rightarrow\ V$, if it is finite rank then it is compact. So, let us take $T\colon\ l_2 \rightarrow l_2$ such that $Tx\ =\ (0,\ x_1, ...,x_n, 0, ...0)$. So, what is T^2? $T^2x\ =\ (0,\ 0,\ x_1,\ ...,\ x_{n-1},\ 0, ...)$

↑ (n+1)

So, $T^{n+1} \equiv 0$, and therefore, it is an important operator. Hence, $\lambda\ =\ 0$ is the only eigenvalue and of course, you can easily find an eigenvector. For instance, e_{n+1} is an eigenvector $T(e_{n+1})\ =\ 0$. You have that $\lambda\ =\ 0$ is the only possible eigenvalue. So, $\sigma(T)\ =\ \{0\}$ and T has finite rank and hence compact.

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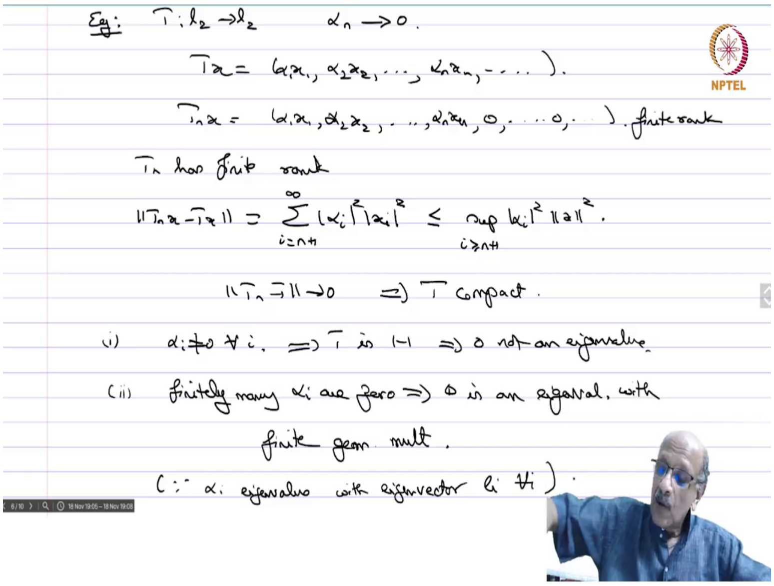

Next, you look at again $T\colon\ l_2 \to l_2$ and let $\{\alpha_n\}$ be a sequence of complex numbers which go to 0 and then you define T in the following way, $Tx = (\alpha_1 x_1,\ \alpha_2 x_2,\ .\ .\ .\ , \alpha_n x_n,\ .\ .)$. And you define $T_n x = (\alpha_1 x_1,\ \alpha_2 x_2,\ .\ .\ .\ , \alpha_n x_n,\ 0,\ 0\ .\ .)$. Then, T_n has finite rank and what can you say about $||T_n x - Tx|| = \sum_{i=n+1}^{\infty} |\alpha_i|^2\, |x_i|^2 \le \sup_{i \ge n+1} |\alpha_i|^2 |x_i|^2$

So, this means that $||T_n x - Tx|| \to 0$ because $\alpha_n \to 0$.

So, sup after some finite stage can be made as small as (())(22:40) because all the α must be less than or equal to ϵ after some finite stage and therefore, this goes to 0. So, T_n has finite rank and T is the limit of finite rank operators. So, this implies that T is compact. So, now, let us see what happens,

1, $\alpha_i \neq 0 \quad \forall\, i$, this implies T is 1-1 implies 0 not an eigenvalue.

2. finitely many $\alpha_i = 0$, implies 0 is an eigenvalue with finite geometric multiplicity, why? Because a_i eigenvalue with eigenvector e_i $\forall\, i$, so that is the reason why you get this.

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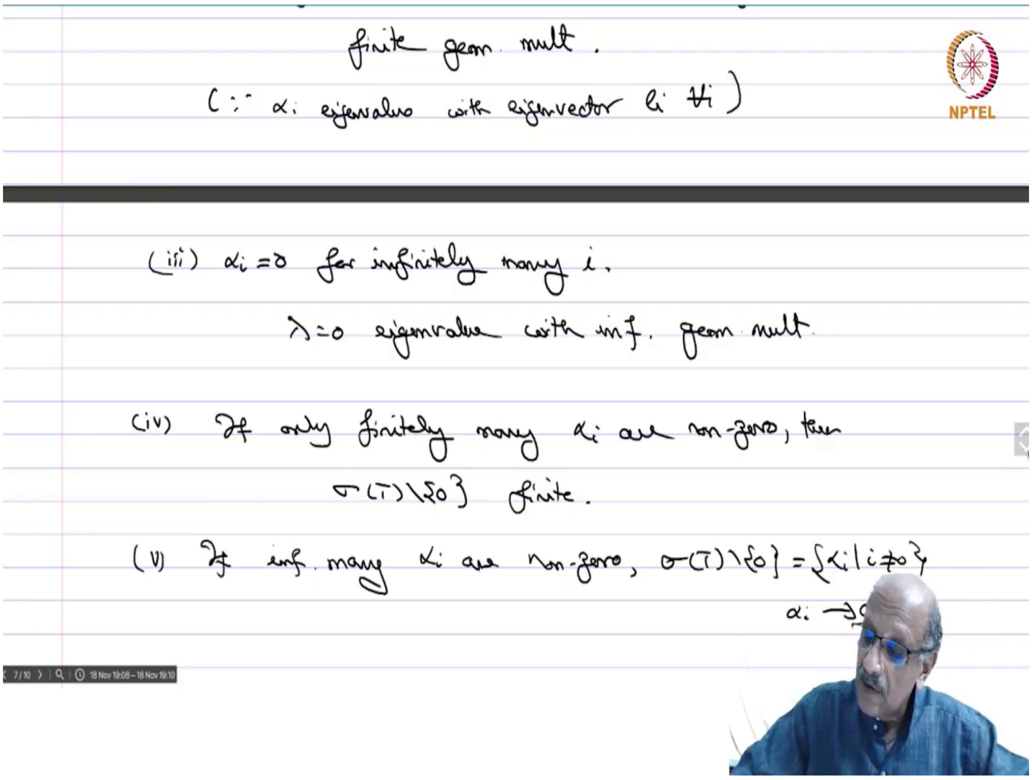

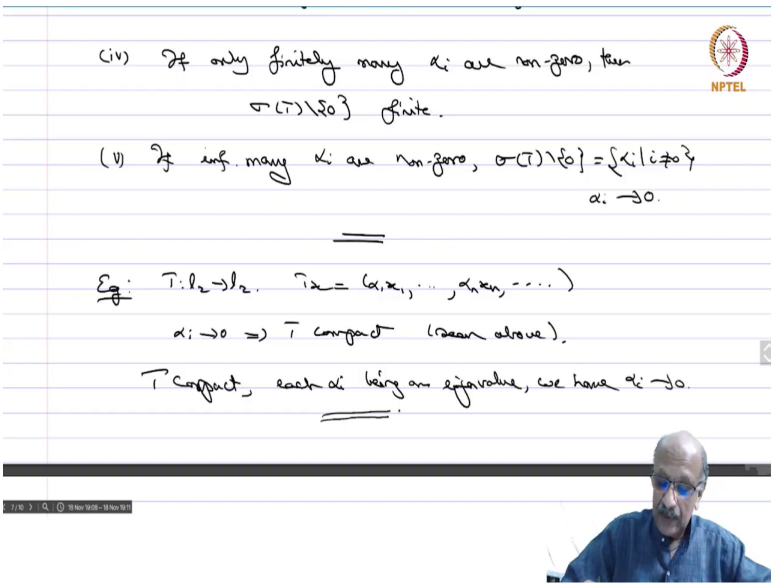

3. $\alpha_i = 0$ for infinitely many i. That means, let us say for all the odd $\alpha_i = 0$ and even $\alpha_i \neq 0$ and go to 0. In that case, you have that $\lambda = 0$ is an eigenvalue with infinite geometric multiplicity, namely null space, has infinite dimension. So, you see for the eigenvalues 0, we can say nothing, it may not be an eigenvalue as we saw in the previous example, it may be the only eigenvalue that is possible or it may not be an eigenvalue as we saw in 1 and it may be an eigenvalue of finite multiplicity or an eigenvalue of infinite multiplicity.

4. if only finitely many α_i are non-zero, then $\sigma(T)\backslash\{0\}$ is finite.

5. if infinitely many α_i are non-zero, then $\sigma(T)\backslash\{0\} = \{\alpha_i / i \neq 0\}$, which is infinite and $\alpha_i \rightarrow 0$ because of our assumption because it gives you a subsequence and therefore, it has to go to 0.

So, all the possibilities in statement (d) of the theorem have been satisfied.

Example. T: $l_2 \rightarrow l_2$ and then you take $Tx = (\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_n x_n, \ldots)$.

So, $\alpha_i \rightarrow 0 \Rightarrow T$ compact, as we saw above.

Now, if T is compact, each α_i being an eigenvalue we have, $\alpha_i \rightarrow 0$. So, it becomes a sequence of eigenvalues, it must converges to 0 and therefore, we see that T is compact if and only if $\alpha_i \rightarrow 0$. So, this is about the general structure of the spectrum of a compact linear operator. Now, we will next look at compact self-adjoint linear operators which are very special and have lots of applications and that we will do next time.