Functional Analysis
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Lecture No. 67
Riesz-Fredholm theory- Part 3

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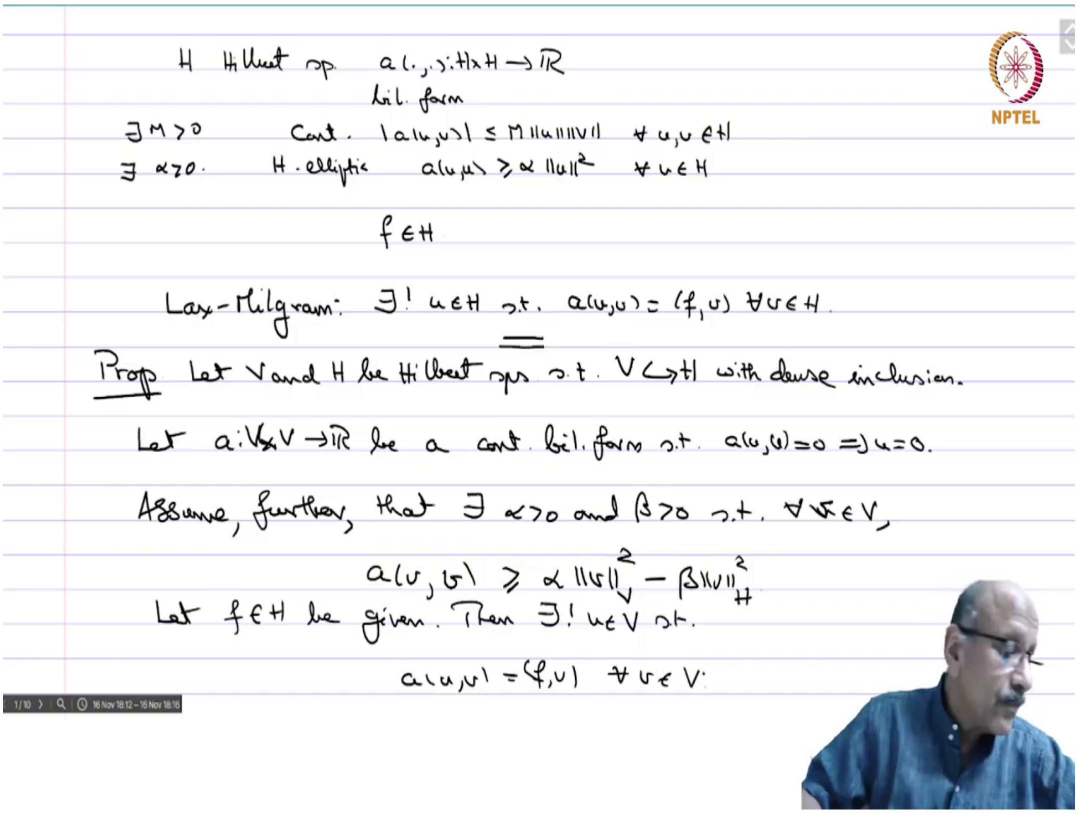

We conclude this section with a theorem, which you can think of as a generalization of the Lax-Milgram lemma. Let me recall the Lax-Milgram lemma. So, H is a Hilbert space and then $a(.\,,\,.): H \times H \rightarrow \mathbb{R}$. I am taking a real Hilbert space. Bilinear form a is continuous that means $\exists\, M > 0$ such that $|a(u,\, v)| \leq M||u||\,||v||\ \forall\, u, v \in H$ and a is H-elliptic that means $\exists\, \alpha > 0$ such that $a(u,\, u) \geq \alpha\, ||u||^2\ \ \forall u \in H$. Given $f \in H$. Then Lax-Milgram says, $\exists$ a unique $u \in H$ such that $a(u,\, v) = (f,\, v)\ \forall v \in H$. So, I recall I hope you remember all those things. Now, the ellipticity condition is always a strong one and it is difficult to have it. So, here is one which relaxes it under some conditions.

Proposition: Let V and H be Hilbert spaces such that V is continuously embedded in H with dense inclusion, it means V is a subspace of H, but the inclusion map is continuous with respect to the respective norms. Assume further that this inclusion is compact.

And then V is a dense subspace of H in the H topology. Let $a(.\,,.): V \times V \to \mathbb{R}$ be a continuous bilinear form such that $a(u, u) = 0 \Rightarrow u = 0$. So, this is all the condition, so this is much weaker than the ellipticity condition is here. Assume further that $\exists\, \alpha > 0$ and $\beta > 0$ such that $\forall v \in V$, we have $a(v, v) \geq \alpha ||v||_V^2 - \beta ||v||_H^2$. So, there is a negative thing, so this is not elliptic at all, but this inequality is good enough. Let $f \in H$ be given, then $\exists$ a unique $u \in V$ such that $a(u, v) = (f, v)_H$ $\forall v \in V$.------------------------(*)

So, this is the same conclusion as the Lax-Milgram Theorem. But you do not have ellipticity. Instead you have a compromise, you have a dense inclusion in the Hilbert space, where you have an inequality of the above form and you also have $a(u, u) = 0 \Rightarrow u = 0$.

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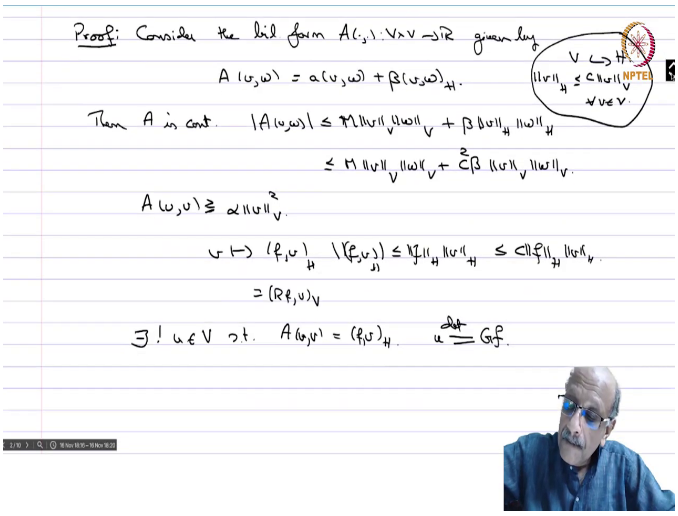

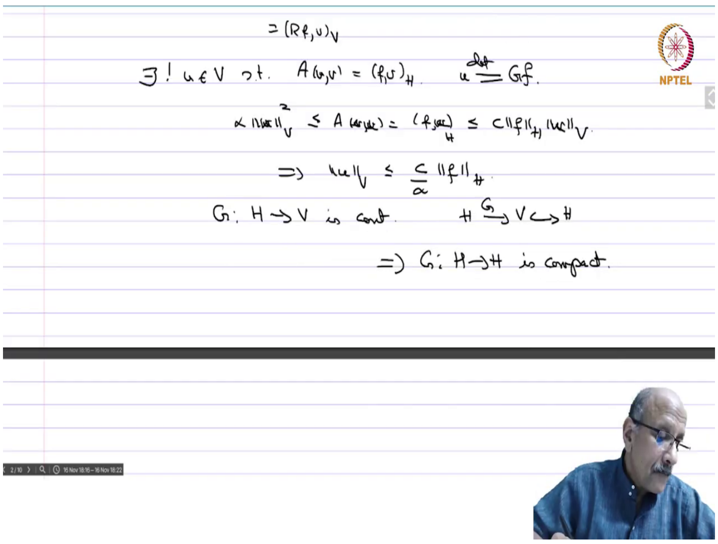

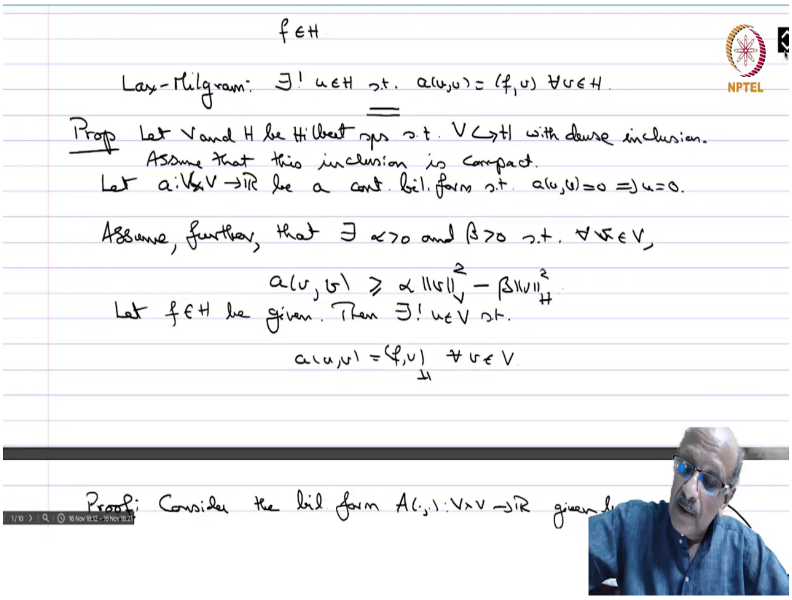

Proof. Consider the bilinear form $A(.\ ,\ .): V \times V \rightarrow \mathbb{R}$ given by

$$A(v,\ w) = a(v,\ w) + \beta(v,\ w)_H.$$

The inner product $(f,\ v)$ obviously is in H because $f \in H$. Then A is continuous, because $|A(v,\ w)| \leq ||M||\ ||v||_V||w||_V + \beta\ ||v||_H\ ||w||_H \leq M\ ||v||_V||w||_V + c^2\ \beta\ ||v||_V||w||_V$, because $||v||_H \leq c\ ||v||_V \quad \forall\ v \in V$ as V is continuously embedded in H. So, A is clearly continuous. What about $A(v,\ v)$? $A(v,\ v) \geq \alpha||v||_V^{\ 2}$. And also, the map $v \mapsto (f,\ v)$ is continuous, because $|(f,\ v)| \leq ||f||_H||v||_H \leq c\ ||f||_H||v||_V$. Therefore, by the Riesz presentation theorem you can write $(f,\ v) = (Rf,\ v)_V$, and therefore, you can apply the Lax-Milgram lemma. There exists a unique $u \in V$ such that $A(u,\ v) = (f,\ v)_H = (Rf,\ v)_V$. therefore, you have this $A(u,\ v) = (Rf,\ v)_V$.

Define $u = Gf$. So, $\alpha||u||_V^{\ 2} \leq A(u,\ u) = (f,\ u)_H = c\ ||f||_H||u||_V$. So, therefore, you get $||u||^2 \leq \frac{c}{\alpha}||f||_H$. Therefore, the mapping $G\colon H \rightarrow V$ is continuous. So, we can now compose, you have a mapping G from H to V and V is included in H and this inclusion is compact. So, this implies that $G\colon H \rightarrow H$ is compact.

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$\Rightarrow G : H \to H$ is compact.

Assume u is a soln. of $(*)$ Then $\forall v \in V$

$$a(u,v) + \beta(u,v)_H = (f+\beta u, v) \quad (**).$$

and conversely.

i.e. $u = G(f+\beta u)$.

$f+\beta u = z \qquad u = Gz$.

$$f = z - \beta u = z - \beta G z$$

u solves $(*)$ $\Leftrightarrow$ $z - \beta G z = f$ where $z = f + \beta u$.

$I - \beta G$ is a compact perturbation of identity in H

It is onto $\Leftrightarrow$ it is 1-1.

$H \to V \hookrightarrow H$

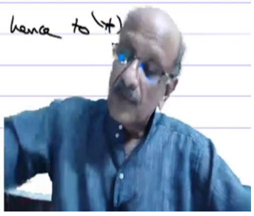
Assume $(I - \beta G)w = 0$. $\Rightarrow w = \beta G w \in V$.

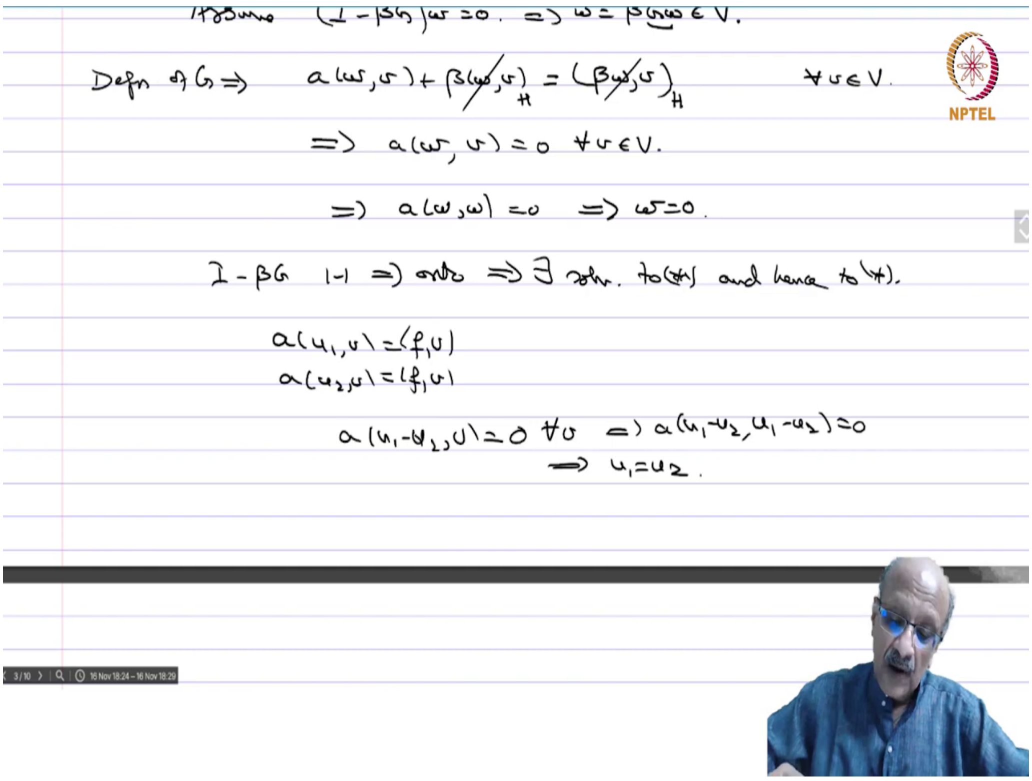
Defn of $G \Rightarrow$ $a(w,v) + \beta(w,v)_H = (\beta w, v)_H \qquad \forall v \in V$.

$\Rightarrow a(w,v) = 0 \ \forall v \in V$.

$\Rightarrow a(w,w) = 0 \Rightarrow w = 0$.

$I - \beta G$ 1-1 $\Rightarrow$ onto $\Rightarrow \exists$ soln. to $(**)$ and hence to $(*)$.





So, now assume u is a solution of (*). Then for every $v \in V$, you have

$a(u, v) + \beta(u, v)_H = (f + \beta u, v)$, -----------(**)

and conversely, if u solve this then $\beta(u, v)_H$ gets cancelled and therefore, $a(u, v) = (f, v)$. So, solving the original problem (*) is the same as solving the problem (**). So $u = G(f + \beta u)$. So, let us write $f + \beta u = z$, so $u = Gz$. Therefore, $f = z - \beta u = z - \beta Gz$. So, u solves (*) if and only if $z - \beta Gz = f$ where $z = f + \beta u$. So, we want to know if you can solve this. So, once you find z then you can say $\frac{z-f}{\beta}$, $\beta > 0$ will give you u which is a solution of the original equation. So, it is enough to solve this equation, where G now is a compact operator from H to H. So, $I - \beta G$ is a compact perturbation of identity in H. So, if you want to solve for any f, so that means, it is onto if and only if it is 1-1, so it is enough to check. So, this is the beauty of this theory. So, if you want to solve any equation you just say, if at all a solution exists then it is unique. So, the uniqueness implies the existence, so that is a nice thing about these results in finite dimensions, that is always true. And now, we are having it in this case. So, you prove uniqueness, that is a much easier thing and that this implies automatically that there exists a solution for any data. So, assume $(I - \beta G)w = 0$. Now, this means what? $w = \beta G w \in V$

Remember $G: H \rightarrow V$ and V is included in H. So, the range is always contained in V. And therefore, Defn. of G implies $a(w, v) + \beta(w, v)_H = (\beta w, v)_H$ for every $v \in V$. G is a Linear map. So, this implies that $a(w, v) = 0 \ \forall v \in V$, w is also in V. This implies $a(w, w) = 0$ and by hypothesis this means that $w = 0$. Therefore, $I - \beta G$ is 1-1, implies onto, implies there exists a solution to (**) and hence to (*). Now, uniqueness is obvious because if you have two solutions $a(u_1, v) = (f, v)$ and $a(u_2, v) = (f, v)$, then $a(u_1 - u_2, v) = 0 \ \ \forall v \in V$ and this implies that $a(u_1 - u_2, u_1 - u_2) = 0$. And therefore, this implies that u_1 has to be equal to u_2. So, you have a unique solution. Now, this Lax-Milgram lemma is very useful as I said in studying the existence of weak solutions to elliptic partial differential equations and so also this one. So, the different kinds of boundary conditions will lead to problems which can be either posed as in the Lax-Milgram framework or in this framework. And therefore, both these results are very useful in the study of PDES.