Functional Analysis
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Lecture No. 66
Riesz-Fredholm Theory – Part 2

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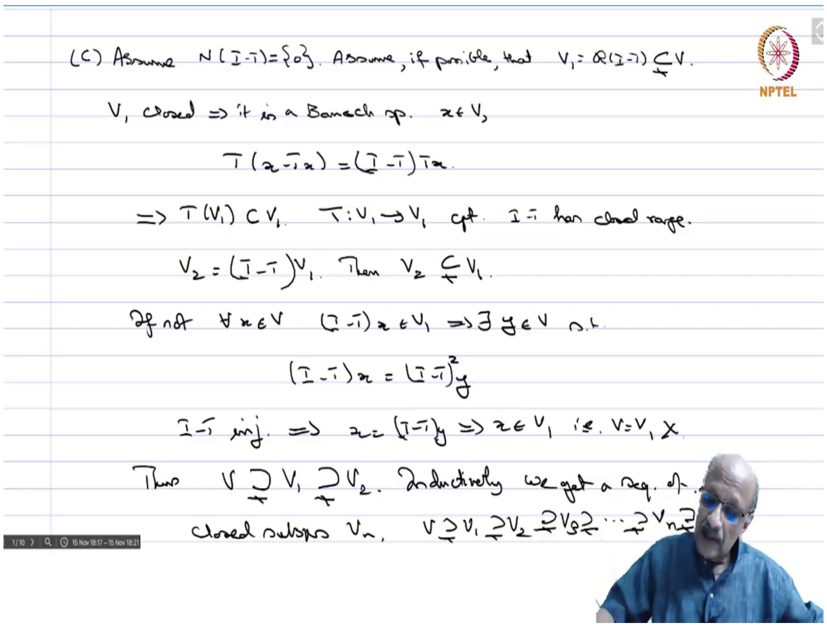

We are in the middle of the proof of the Fredholm alternative, so we have shown that if T is a compact perturbation of identity then the $N(I - T)$ is finite dimensional and $R(I - T)$ is closed. Now, we come to the important property which is really a finite dimensional property namely $I - T$ is 1-1 if and only if it is onto.

This is part (c). Assume $N(I - T) = \{0\}$ that is $I - T$ is injective. Assume if possible that $V_1 = R(I - T) \subset V$. So, $V_1 \neq V$, we have to get a contradiction. So, V_1 is closed, implies it is a Banach space in its own right, if $x \in V$, then you have $T(x - Tx) = (I - T)Tx$. So T of something in V_1 is again in V_1 because this again in the $R(I - T)$, so this implies $T(V_1) \subset V_1$ and V_1 is a Banach space, it is compact and therefore it also has closed range, therefore $I - T$ has a closed range. Let $V_2 = (I - T)V_1$, then V_2 will also be strictly contained in V_1, i.e., $V_2 \subset V_1$, if not $\forall\, x \in V$, $(I - T)x \in V_1$ and if you assume that $V_1 = V_2$ implies $\exists\, y \in V$ such that $(I - T)x = (I - T)^2 y$, because $(I - T)x \in V_1$, it should be the image of something, it should be in V_2, V_2 is nothing but $(I - T)^2 y$. And then $(I - T)$ is injective implies that $x = (I - T)y \Rightarrow x \in V_1$ i.e., $V = V_1$. This is a contradiction, because we are assuming V is strictly bigger than V_1, thus $V \supset V_1 \supset V_2$. So, inductively we proceed like this, and get a sequence of closed subspaces V_n and you have $V \supset V_1 \supset V_2 \supset \ldots \supset V_n \supset \ldots$ this should not end because at any stage if it ends then you will get $V_1 = V_n$.

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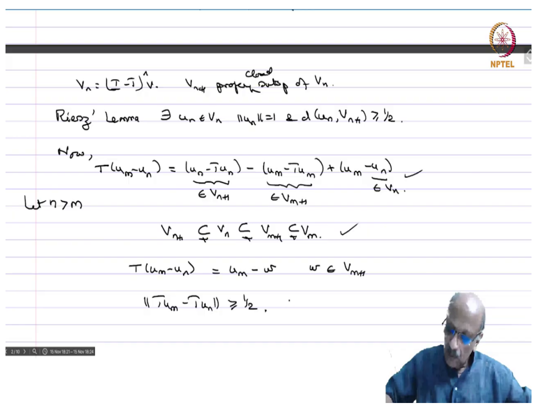

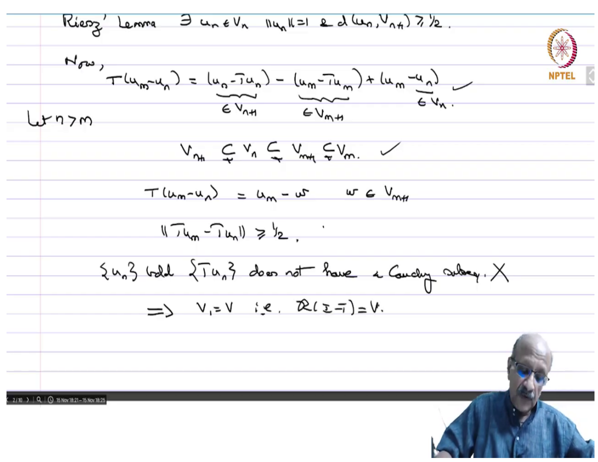

So, we have $V_n = (I - T)^n V$. V_{n+1} proper closed subspace of V_n. By Riesz lemma, there exists $u_n \in V_n$ such that $||u_n|| = 1$ and $d(u_n, V_{n+1}) \geq 1 - \epsilon = 1/2$ taking $\epsilon = 1/2$. Now $T(u_m - u_n) = (u_n - T u_n) - (u_m - T u_m) + (u_m - u_n)$, let $n > m$.

Note, $(u_n - T u_n) \in V_{n+1}$,

$(u_m - T u_m) \in V_{m+1}$, $\quad V_{n+1} \subset V_n \subset V_{m+1} \subset V_m$, therefore $(u_m - u_n) \in V_m$.

Therefore you can write $T(u_m - u_n) = u_m - w, \;\; w \in V_{m+1}$. If you look at where all it belongs and these inclusions and therefore so from these two it follows that $w \in V_{m+1}$. Therefore, $||Tu_m - Tu_n|| \;\geq\; \frac{1}{2}$. What does this imply? $\{u_n\}$ is bounded as $||u_n|| = 1$ and $\{Tu_n\}$ does not have a Cauchy subsequence, which is a contradiction to the fact that T is compact. And therefore, this implies that $V_1 = V$ that is $R(I - T) = V$. So, if it is 1-1 then it is in fact onto. So, now we have to claim the other one.

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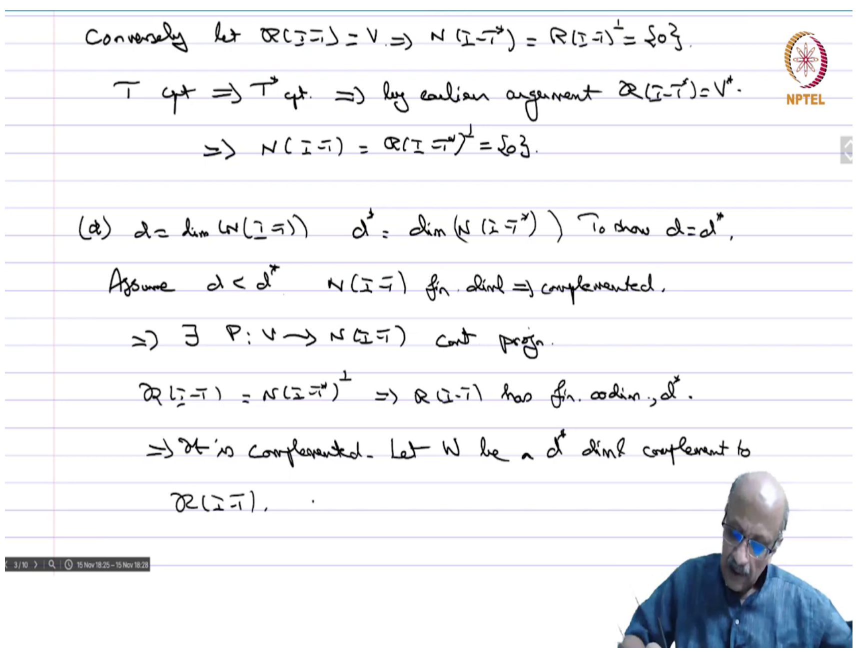

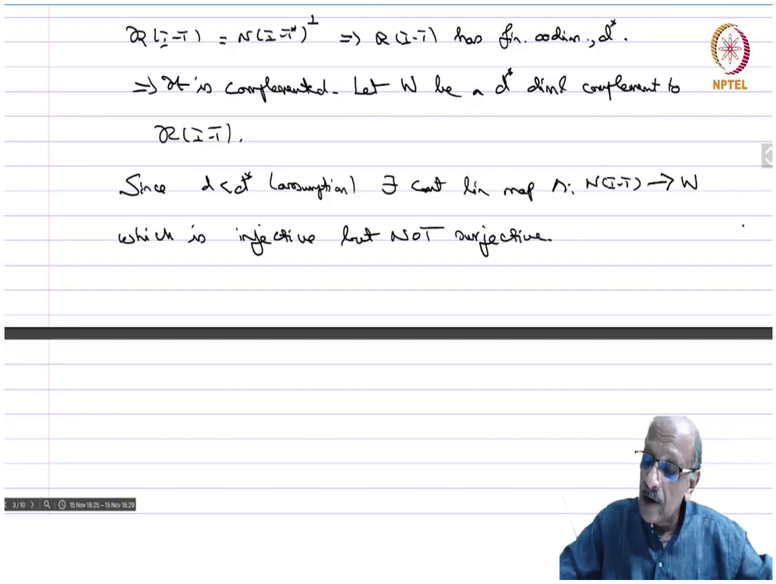

Conversely let $R(I - T) = V$, so it is onto we have to show that it is one-one. Then $N(I - T^*) = R(I - T)^{\perp} = V^{\perp} = \{0\}$. Now T compact we know this implies that T^* is compact and it is 1 to 1 and therefore by earlier argument we have $R(I - T^*) = V^*$. And this implies that $N(I - T) = R(I - T^*)^{\perp} = \{0\}$. So, that completes the proof of this part.

Now, for the last part, so we want to show that $d = \dim(N(I - T))$ and $d^* = \dim(N(I - T^*))$ to show $d = d^*$. Assume $d < d^*$, so $N(I - T)$ is finite dimensional implies complemented implies there exists a continuous projection $P\colon V \to N(I - T)$. Now, $R(I - T)$ is $N(I - T^*)^{\perp}$, so this implies that $R(I - T)$ has finite codimension d^*. And therefore it is complemented again we have seen these things before, so let W be d^*dimensional complement to $R(I - T)$. Now, since $d < d^*$ (assumption) there exists a continuous linear map $\Lambda : N(I - T) \to W$. These are both finite dimensional spaces so you can see that any linear map is continuous, which is injective but not subjective. You cannot have a subjective map because W has smaller dimension than $N(I - T)$ and therefore there you can have a map which is injective but not subjective.

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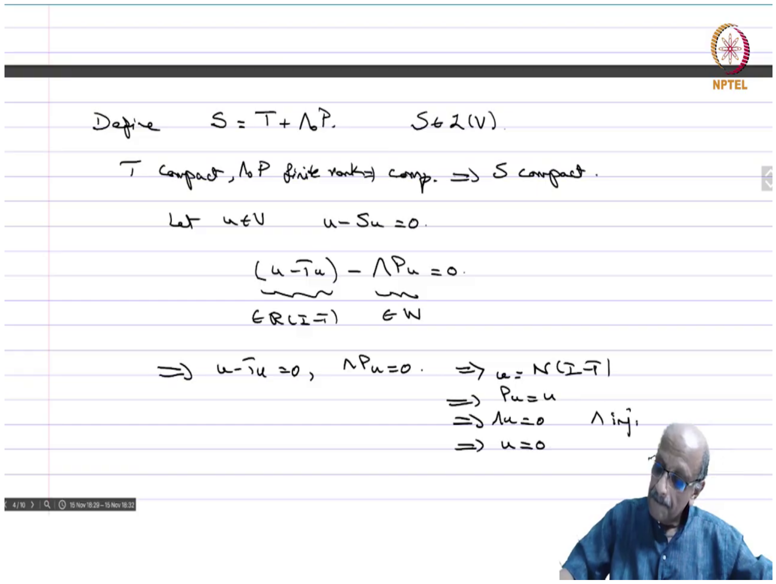

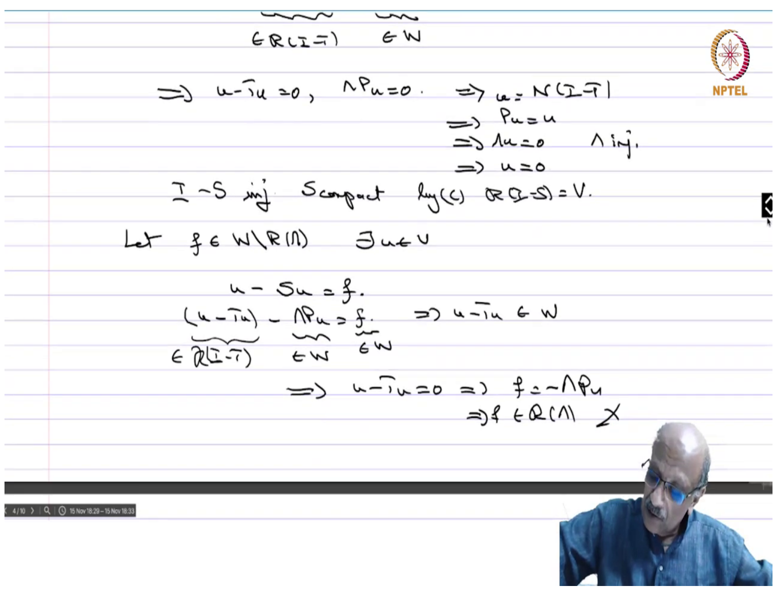

Define $S = T + \Lambda \circ P$. So, T is a map from V into itself, P goes from V to the null space and Λ goes from null space into W and therefore $S \in L(V)$. And T is compact and $\Lambda \circ P$ is of finite rank because its range is inside W which is finite dimensional and therefore compact. Therefore, sum of compact operators is compact, so S is also compact. Let u be such that $u - Su = 0$. If you expand that $(u - Tu) - \Lambda P u = 0$, $(u - Tu) \in R(I - T)$ and $\Lambda P u \in W$ which is the complement of $R(I - T)$ and therefore this is a direct sum therefore you have written the 0 vector as a sum of one in $R(I - T)$ and other in W, so each of them has to be 0. This implies that $u - Tu = 0$ and $\Lambda P u = 0$. So, this implies that $u \in N(I - T)$, so this implies that

$Pu = u \Rightarrow \Lambda u = 0 \Rightarrow u = 0$ because Λ is injective. And consequently, you have that $I - S$ is injective, S compact, so by (c) $R(I - S) = V$. So, let $f \in W \setminus R(\Lambda)$ as we know that Λ is not subjective into W and therefore you can find an f which is not in $R(\Lambda)$. So, now it is. $I - S$ is onto so there exists a $u \in V$ such that $u - Su = f$. So, once again let us write that $(u - Tu) - \Lambda Pu = f$. Again, $(u - Tu) \in R(I - T)$ and $\Lambda Pu \in W$ and therefore again by the direct sum decomposition you have that now $f \in W$ this implies that $(u - Tu) \in W$ and it also belongs to the $R(I - T)$ and they are complementary spaces this implies that $(u - Tu) = 0$ and this implies that $f =- \Lambda Pu \Rightarrow f \in R(\Lambda)$ but that is a contradiction because we have chosen f not to be in the range. So, we have a contradiction all over and therefore and that started from the fact that $d < d^*$.

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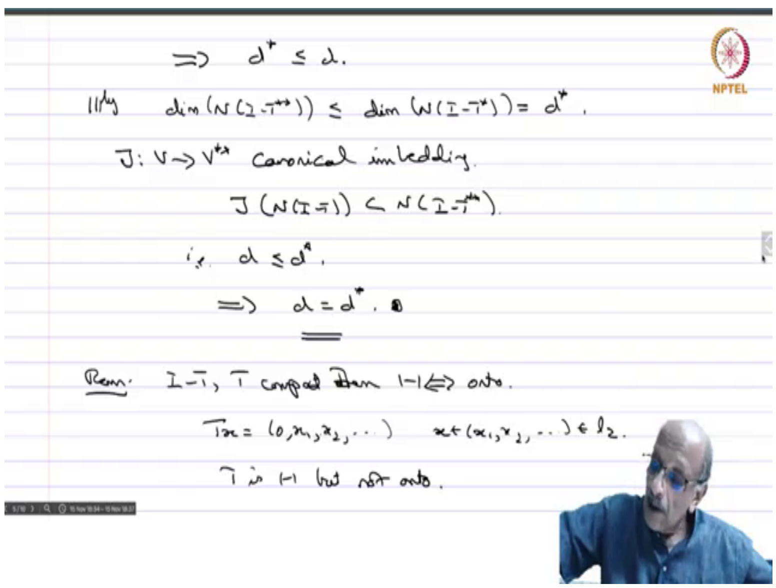

So, this implies that $d^* \leq d$. Now, similarly $\dim(N(I - T^{**})) \leq \dim(N(I - T^*)) = d^*$. Now, if $J : V \rightarrow V^{**}$ is the canonical embedding, then $J(N(I - T)) \subset N(I - T^{**})$. This very trivial checking all you have to do is to find $(I - T^{**})\, Jx$ and then apply the definition of T^{**} which and use the adjoint equation every time and then you will get that $N(I - T^{**})$={0}. Therefore J is isometric isomorphism so the dimension of the space $J(N(I - T))$ is precisely d i.e., $d \leq d^*$. And therefore, this implies the $d = d^*$ and the theorem is completely proved.

Remark. We proved that $I - T$, T compact, implies, one-to-one if and only if onto. But this is not generally true for any operator in the Hilbert space as you have already seen you take $Tx = (0, x_1, x_2, \ldots)$, $x = (x_1, x_2, \ldots)$, then T is one-to-one but not onto.

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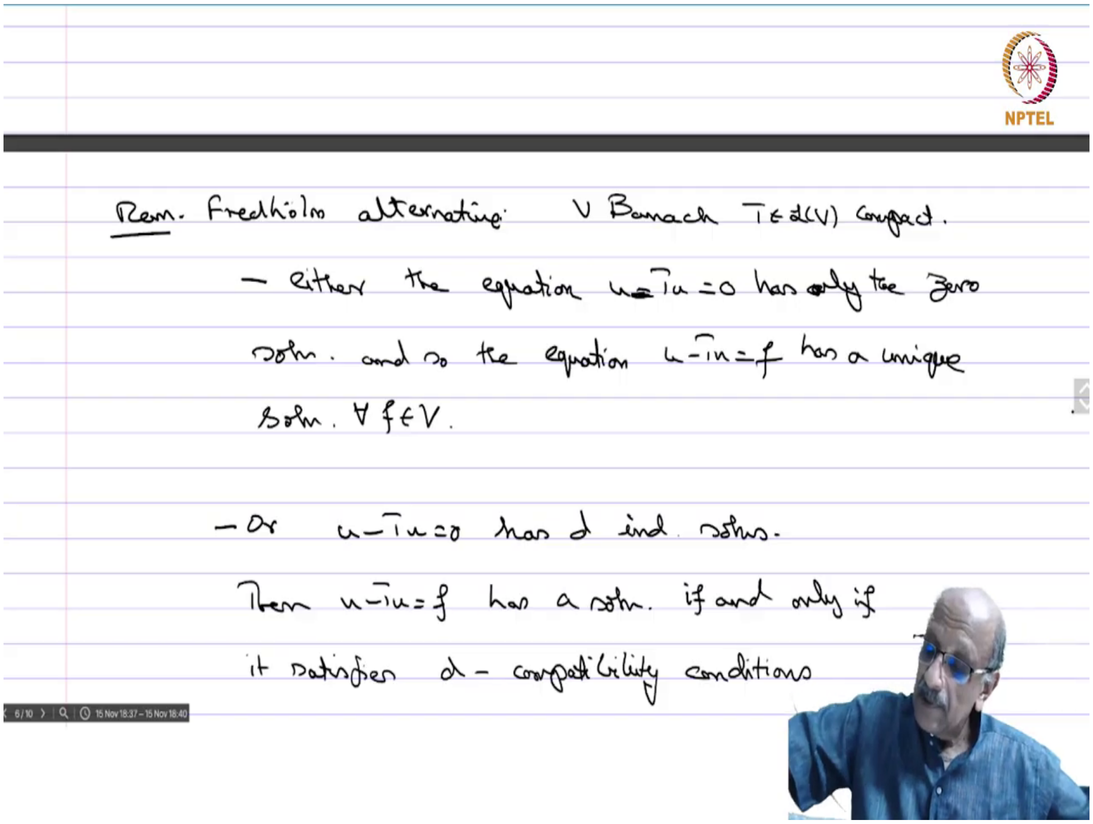

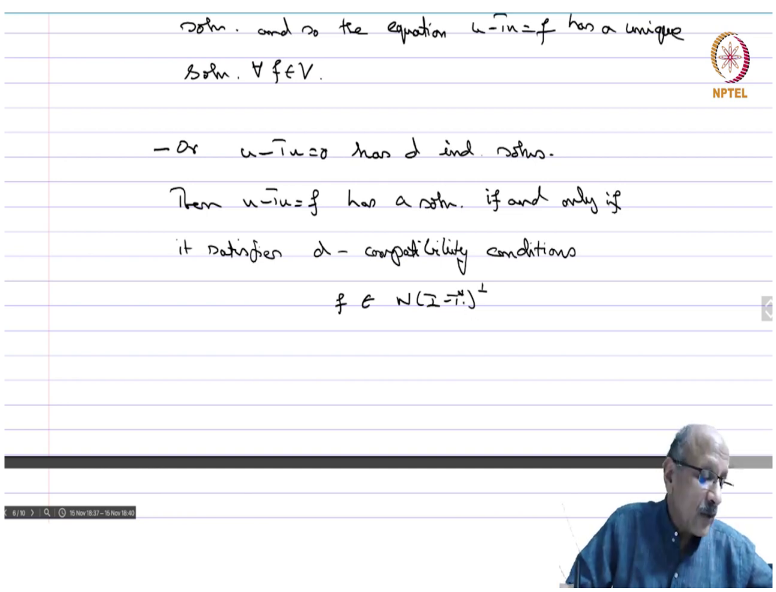

So, in finite dimensional spaces we also always have that, see in finite dimensional spaces is difficult to talk about the null space, dimension of the null space whereas the dimension of the range is nothing but the number of linearly independent columns and the dimension of $R(T^*)$ is nothing but the number of linearly independent rows. So, you can say that the row rank is the column rank, But then, by the rank nullity theorem we automatically get that the dimensions of the null space also coincide. Now, in infinite dimensions it is difficult to talk about the dimension of the range, because they are all infinite-dimensional, so there is no point in saying they are equal, they are both infinity, so where as the null spaces are finite dimensional and that is why we state the theorem in this fashion.

Remark: The Fredholm alternative, why do you call it an alternative? We saw V Banach space and $T \in L(V)$ is compact. So, one of the alternatives holds either the equation $u - Tu = 0$ has only the 0 solution and so the equation $u - Tu = f$ has a unique solution for every $f \in V$, so this is one alternative; or $u - Tu = 0$ has d independent solutions, then $u - Tu = f$ has a solution if and only if it satisfies d-compatibility conditions namely $f \in N(I - T^*)^{\perp}$. So, this is a d-dimensional space so it is being in the orthogonal means it should vanish on each one of the basis elements here therefore it will vanish over the entire space and therefore you have the f must be in this it should satisfy d-compatibility conditions. So, that is the reason why you have this.

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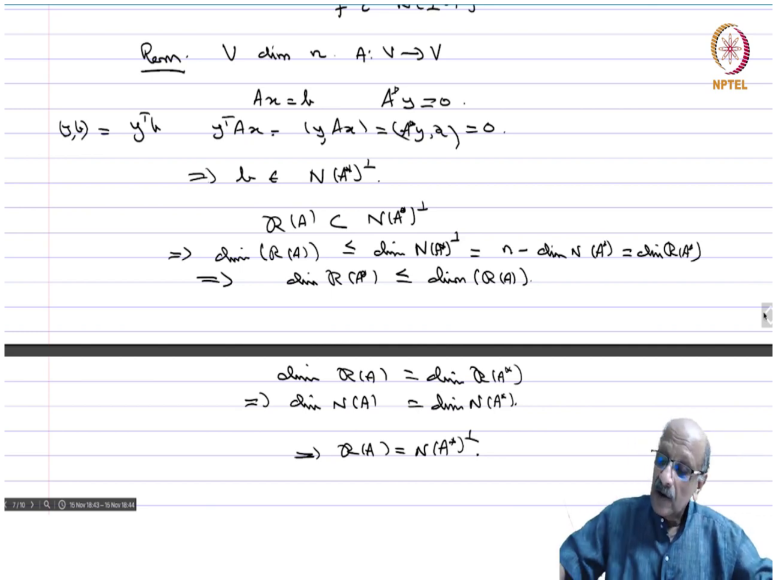

Remark: In finite dimensions, for instance, if you have dim $V = n$, so you have A: $V \to V$ some matrix which generates an operator, let us work with the real space for the or does not matter what we working with. So, we want to look for solution of $Ax = b$, to solve n linear equations in n unknown, (actually you can do it even for rectangular matrices when $V \to W$ it does not matter, we will do it in this case). so let us assume that $A^* y = 0$. $(y, b) = y^T b$.

Then $y^T Ax = (y, Ax) = (A^* y, x) = 0$, this means that if you have a solution to $Ax = b$, then a necessary condition is that $b \in N(A^*)^{\perp}$. $R(A) \subset N(A^*)^{\perp}$, so this means that the $\dim R(A) \leq \dim N(A^*)^{\perp} = n - \dim N(A^*) = \dim R(A^*)$. So, dim $(R(A)) \leq$ dim $R(A^*)$ and if you apply it to A^*, you have dim $(R(A^*)) \leq \dim(R(A^{**}))$ but, A^{**} is the same as A. And therefore, you have this and precisely, you have dim $(R(A)) = \dim(R(A^*))$. And this is saying the row rank is the same as column rank in the real case, in the complex case it says same column rank of the conjugate transpose, but conjugate transpose and transpose have the same column rank and therefore its dimension, so row rank equals column rank. That is this theorem. And also by the rank nullity theorem this shows the $\dim(N(A)) = \dim(N(A^*))$. And we have also proved because you have $R(A) \subseteq N(A^*)^{\perp}$ and both have the same dimension and therefore this also implies that $R(A) = N(A^*)^{\perp}$. So, all the theorems we proved in just two lines using these arguments.

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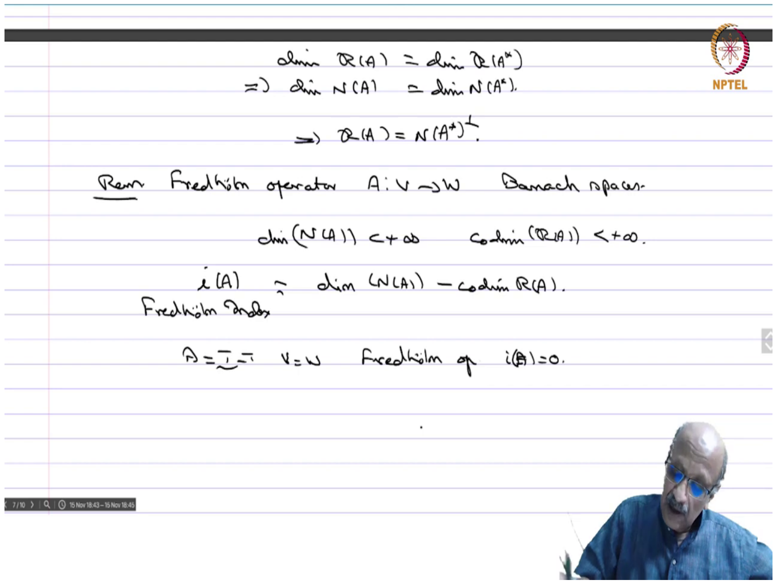

Remark. Riesz-Fredholm Theory is the starting point of the study of Fredholm operator, so Fredholm operator $A: V \rightarrow W$ Banach spaces, is such that $\dim N(A) < \infty$, co-dimension $(R(A)) < \infty$, then it is called Fredholm operator and you call $i(A) = \dim N(A) -$ co-dimension$(R(A))$ is the Fredholm index. If $A = I - T$ compact perturbation of identity, $V = W$ then it is a Fredholm operator with, $i(A) = 0$. So, that is the theorem which we have proved. So, this is a very important concept in operator theory.