Functional Analysis
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Lecture No. 65
Riesz-Fredholm Theory – Part 1

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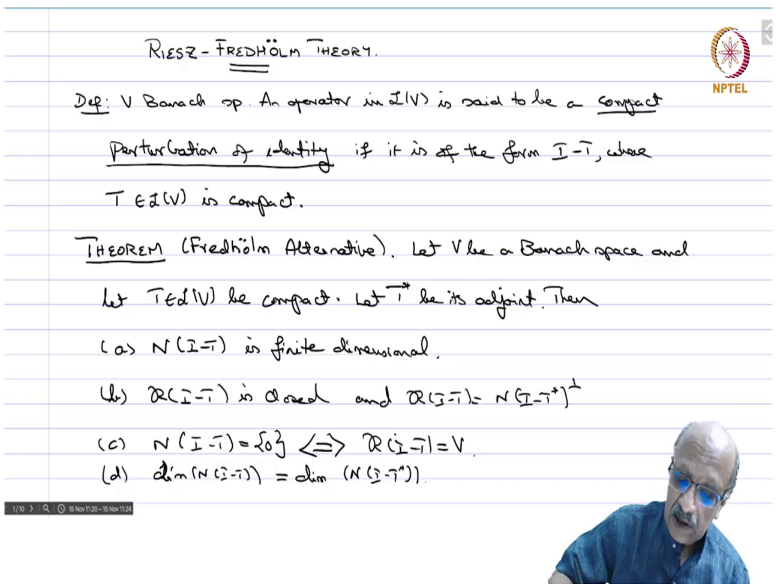

We will now prove a very important theorem, where we are going to imitate many things, many properties of finite dimensional operators. This is called the Riesz-Fredhölm theory.

Definition: V Banach space, an operator in $L(V)$ is said to be compact perturbation of identity if it is of the form $I - T$, where $T \in L(V)$ is compact, such operators are very important because they help us to study the spectrum of compact operators on one hand and secondly they imitate a lot of the properties of finite dimensional operators.

So, we have this important theorem which we will prove slowly.

Theorem: (Fredhölm alternative) I have already used this terminology earlier and we will comment on it and at the end of the proof of this theorem. So, let V be a Banach space and let $T \in L(V)$ be compact, let T^* be its adjoint, then

(a) $N(I - T)$ is finite dimensional,

(b) $R(I - T)$ is closed and $R(I - T) = N(I - T^*)^{\perp}$.

(c) $N(I - T) = \{0\} \Leftrightarrow R(I - T) = V$. So $N(I - T)$ is 0 means it is an injective map or 1-1 and $R(I - T) = V$ means it is surjective, so this is exactly like finite dimensions $I - T$ is 1-1 if and only if it is onto. Now, in finite dimensions we have also this, if you took the rank nullity theorem in to counter then it will be $R(I - T) = R(I - T^*)$, which is the famous statement that the row rank of a matrix is the column rank, so this is just the infinite dimensional analogue of it.

(d) dim $(N(I - T))$= dim $N(I - T^*)$.

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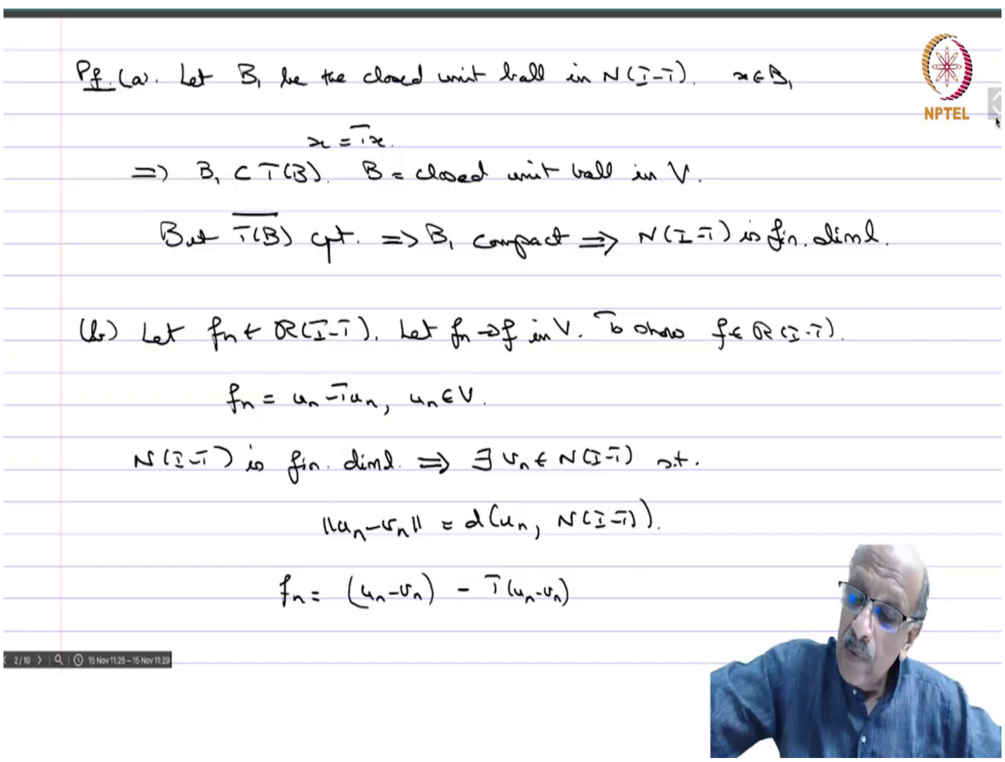

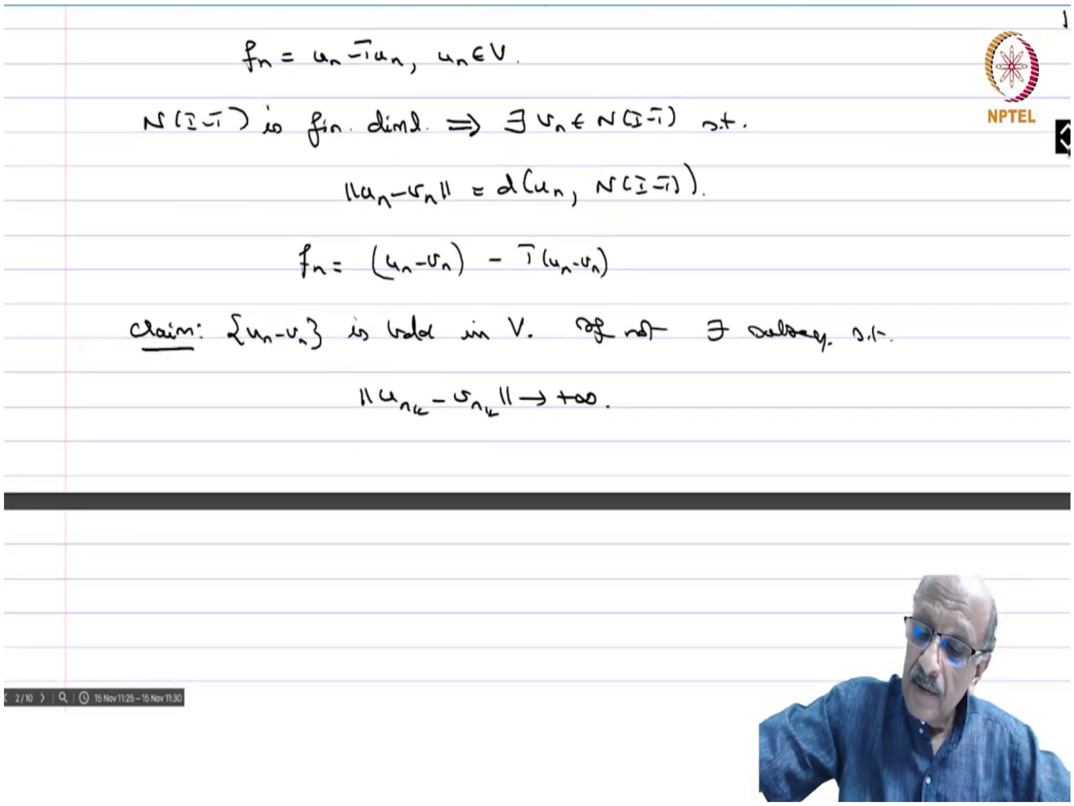

Proof. (a) Let B_1 be the closed unit ball in $N(I - T)$, $N(I - T)$ is closed so it is a Banach space in its own right and B_1 is the closed unit ball, so if $x \in B_1$, then $x = Tx$, so $B_1 \subset T(B)$, where B is the closed unit ball in V, after all $N(I - T)$ is a subspace of V, so B_1 itself is a subset of B. But $\overline{T(B)}$ is compact and B_1 is a closed subspace of a compact set and therefore implies B_1 is compact and if you have that the closed unit ball is compact, the space must be finite dimensional, so $N(I - T)$ is finite dimensional.

(b) so we have to show $R(I - T)$ is closed and of course the second part of it is easy because we have already seen if you have a closed range then it is equal to the annihilator of the null space of the adjoint, so that part is well known. Let $f_n \in R(I - T)$, let $f_n \rightarrow f$ in V, so to show $f \in R(I - T)$, so $f_n = u_n - Tu_n$, $u_n \in V$ is in the range, now $N(I - T)$ is finite dimensional, so this implies that there exists $v_n \in N(I - T)$ such that $|| u_n - v_n|| = d(u_n, N(I - T))$. Why is this so? Now, what is $d(u_n, N(I - T))$? It is nothing but the infimum of $||u_n - v|| \ \forall v \in N(I - T)$. So, you take a sequence say $\{v_m\}$ which converges to $\{u_n - v_m\}$ converges to the distance, but then you have the infinite dimensional space, therefore a bounded sequence has a convergent subsequence and therefore the limit of the convergent subsequence will be in fact the point that always the minimize the distance, this minimizer is realized, and therefore if you have a finite dimensional space, you can always find a v_n such that $u_n - v_n$ is this. So, I have already mentioned this solution, so please write down the arguments so that you will be clear about it. So, you take a minimizing sequence and it will have a convergent subsequence and that limit should be precisely the element which you are looking for. So, you can have this and therefore we can write $f_n = (u_n - v_n) - T(u_n - v_n)$.

Claim: $\{u_n - v_n\}$ is bounded in V, so it is a bounded sequence. If not there exists a subsequence such that $||u_{n_k} - v_{n_k}|| \to + \infty$.

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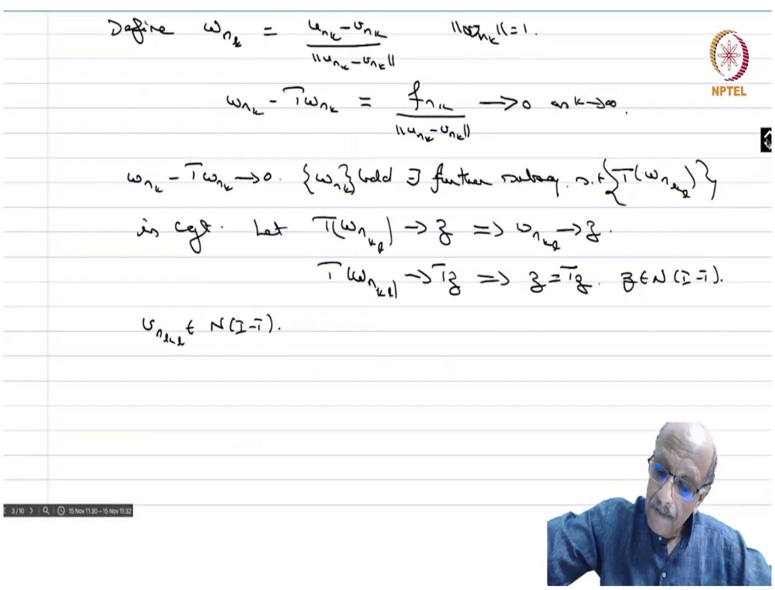

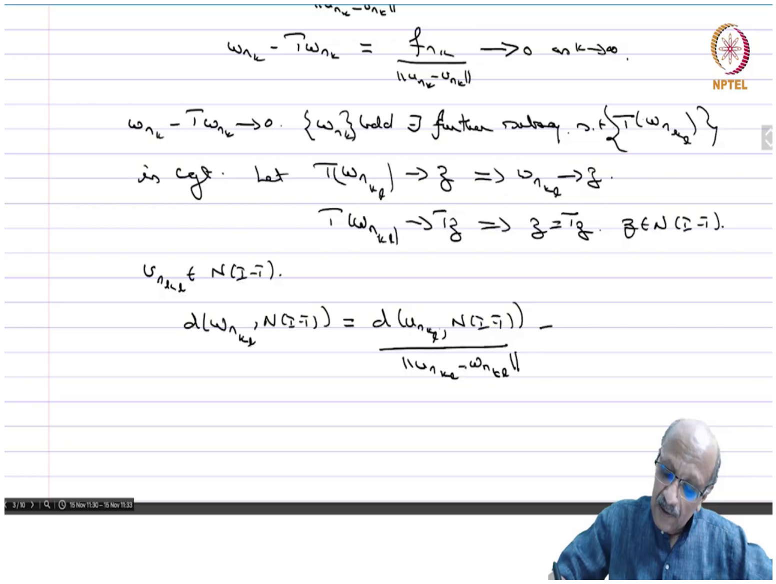

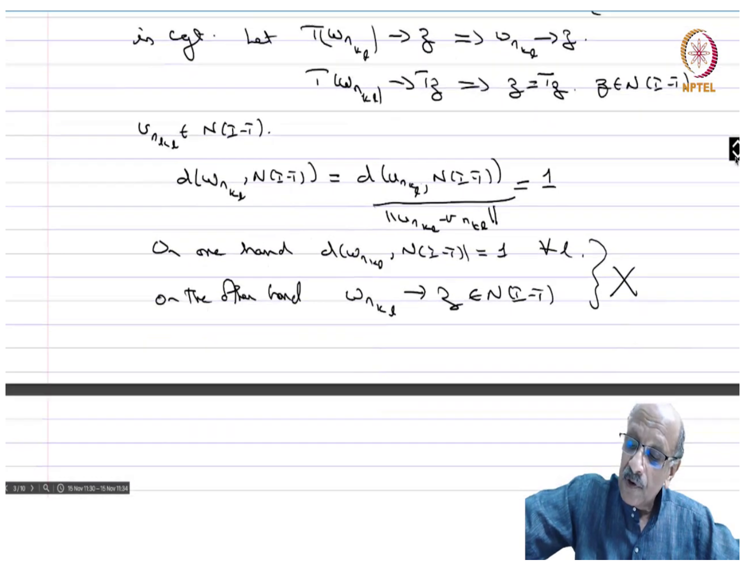

Define $w_{n_k} = \frac{u_{n_k} - v_{n_k}}{\left\| u_{n_k} - v_{n_k} \right\|}$. So, $\left\| w_{n_k} \right\| = 1$ and $w_{n_k} - T\left(w_{n_k}\right) = \frac{f_{n_k}}{\left\| u_{n_k} - v_{n_k} \right\|}$ is a convergent sequence, so it is bounded and divided by something going to infinity, so $w_{n_k} - T\left(w_{n_k}\right) \to 0$ as $k \to \infty$. Now, $\{w_{n_k}\}$ is bounded because it has norm 1, so there exists a further subsequence $\left(w_{n_{k_l}}\right)$ such that $T\left(w_{n_{k_l}}\right)$ is convergent. So, let $T\left(w_{n_{k_l}}\right) \to z \Rightarrow w_{n_{k_l}} \to z$. Then $T\left(w_{n_{k_l}}\right) \to T\,z$, and therefore this implies $z = Tz$, $z \in N(I - T)$.

Now, $v_{n_{k_l}} \in N(I - T)$ and therefore, $d\left(w_{n_{k_l}},\ N(I - T)\right) = \dfrac{d\left(u_{n_{k_l}},\ N(I-T)\right)}{\left\|u_{n_{k_l}} - v_{n_{k_l}}\right\|} = 1$. So, on one hand $d\left(w_{n_{k_l}},\ N(I - T)\right) = 1\ \forall\, l$ and on the other hand $w_{n_{k_l}} \to z \in N(I - T)$ and that is a contradiction.

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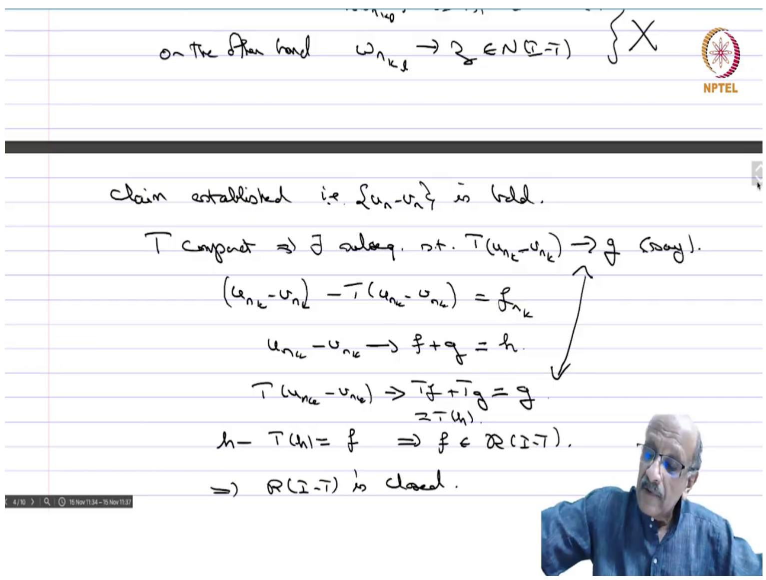

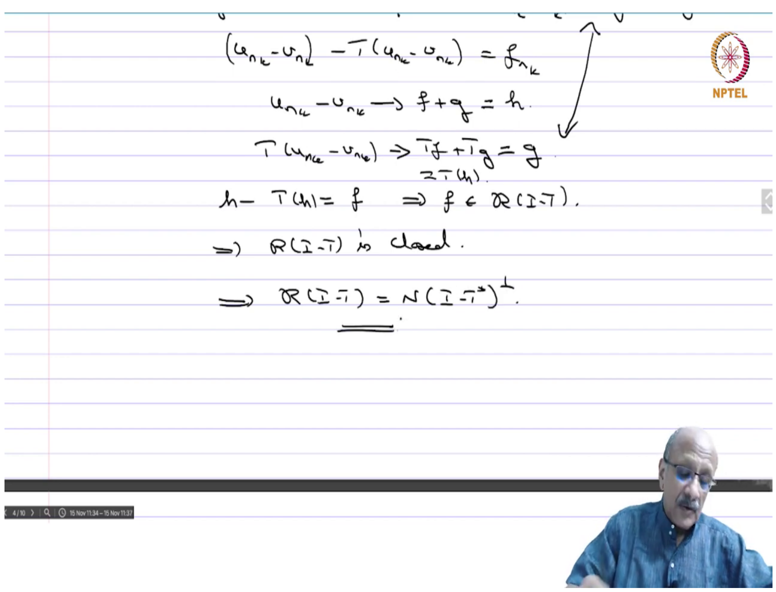

And therefore, claim is established that $\{u_n - v_n\}$ is bounded. So, T is compact implies $\exists$ subsequence such that $T(u_{n_k} - v_{n_k}) \rightarrow g$. Because T is compact, so given a bounded sequence you have a convergent subsequence. so $(u_{n_k} - v_{n_k}) - T(u_{n_k} - v_{n_k}) = f_{n_k}$. So, $f_{n_k} \rightarrow f$ $T(u_{n_k} - v_{n_k}) \rightarrow g$. So, $(u_{n_k} - v_{n_k}) \rightarrow f + g = h$. So, $T(u_{n_k} - v_{n_k}) \rightarrow Tf + Tg = g$, let just call $Tf + Tg = Th$, so, $Th = g$. So, $h - Th = f + g - g = f$. And therefore, this says that R$(I - T)$. Therefore, $R(I - T)$ is closed. And once you have that that automatically implies that R$(I - T) = N(I - T^*)^{\perp}$. So, next we will take up the other part (())(17:02).