Functional Analysis
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Lecture No. 64
Compact operators – Part 2

We saw the limit of compact operators is compact, and since every operator with finite rank is compact, the limit of operators with a finite rank is compact. Now, the question we wanted to ask "is the limit of any compact operator of finite rank?" That is not generally true. So, but here is a situation, where it is true:

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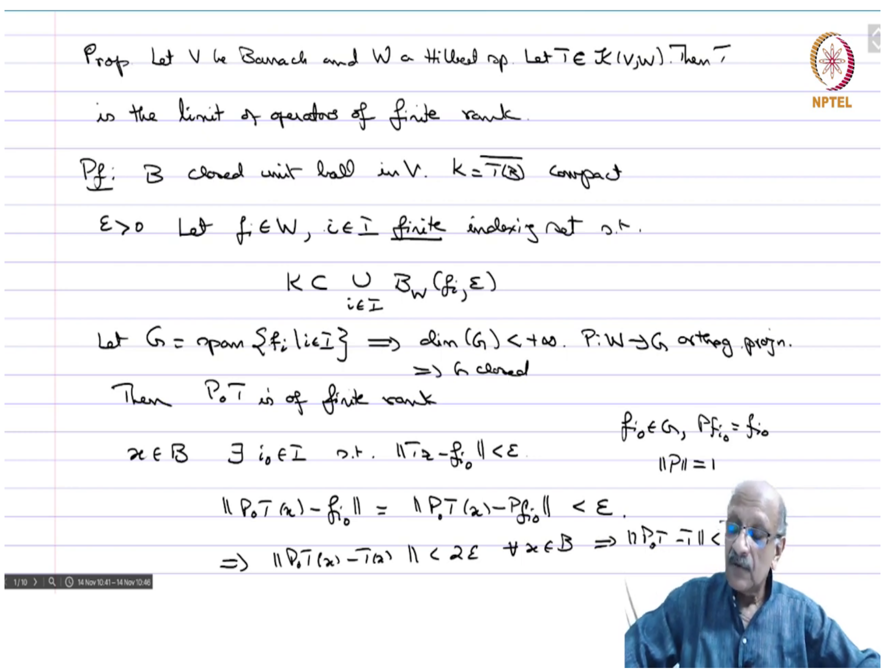

Proposition. Let V be Banach and W be a Hilbert space. Let $T \in \mathcal{K}(V, W)$ then T is the limit of operators of finite rank.

Proof: B closed unit ball in V and $K = \overline{T(B)}$ is compact. Let $\epsilon > 0$ and let $f \in W$, I finite indexing set, such that $K \subset \bigcup_{i \in I} B_w(f_i, \epsilon)$. Let $G = span\{f_i / i \in I\}$. We do not know about its linear independence, but this implies that dim (G) is finite. Let $P : W \to G$ be the orthogonal projection. So, dim $(G) < \infty \Rightarrow G$ is closed and $P : W \to G$ is the orthogonal projection. Then $P \circ T$ is of finite rank. Because T takes V to W and P takes W to G. G is a finite dimension space therefore $P \circ T$ is of finite rank. Now, let $x \in B$ then there exists $i_0 \in I$ such that $||Tx - f_{i_0}|| \leq \epsilon$. Then $|| P \circ T(x) - f_{i_0}|| = || P \circ T(x) - Pf_{i_0}||$ as $f_{i_0} \in G$, therefore $Pf_{i_0} = f_{i_0}$. So, P being an orthogonal projection $||P||$ is equal to 1. And therefore $|| P \circ Tx - f_{i_0}|| = || P \circ Tx - Pf_{i_0}|| < \epsilon$ because $||P|| = 1$. Normally a projection will have norm ≥ 1. And here we have an orthogonal projection. Therefore it is a unit. So this implies that $|| P \circ T(x) - T(x)|| < 2\epsilon \; \forall x \in B$ and this implies that $|| P \circ T - T || < 2\epsilon$. So, this completes the proof.

So, we can approximate every compact operator for any given ϵ by an operator of finite rank. And therefore you have that it is the limit of operators of finite rank.

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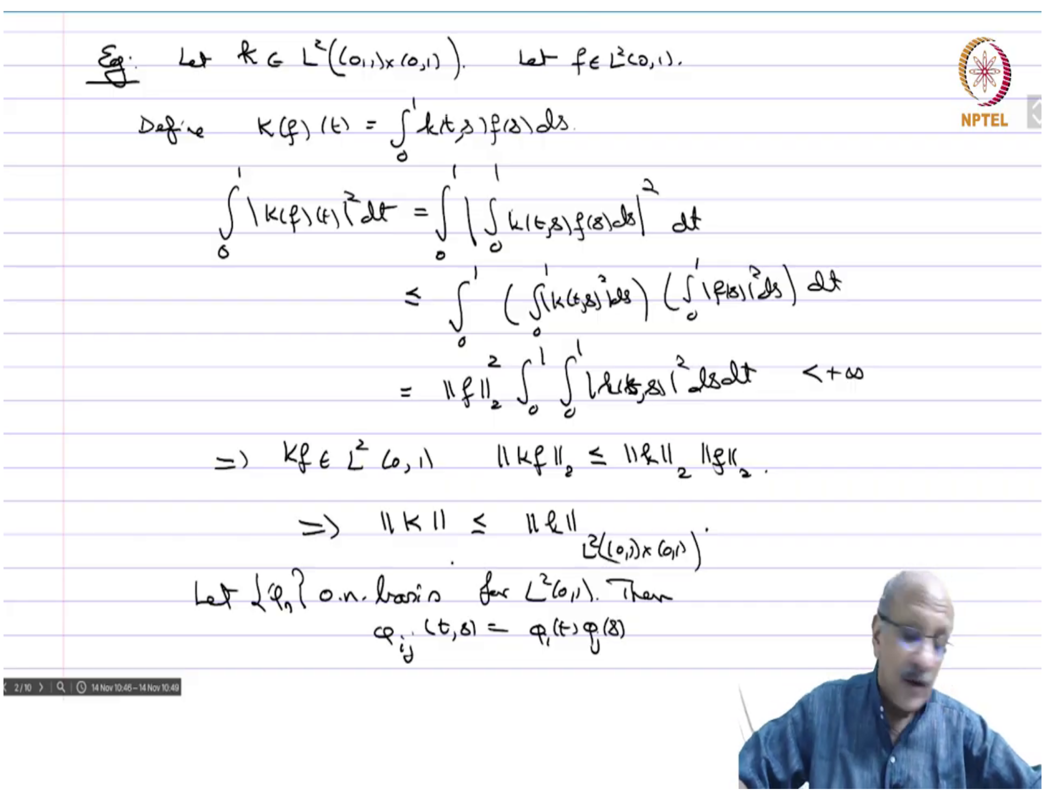

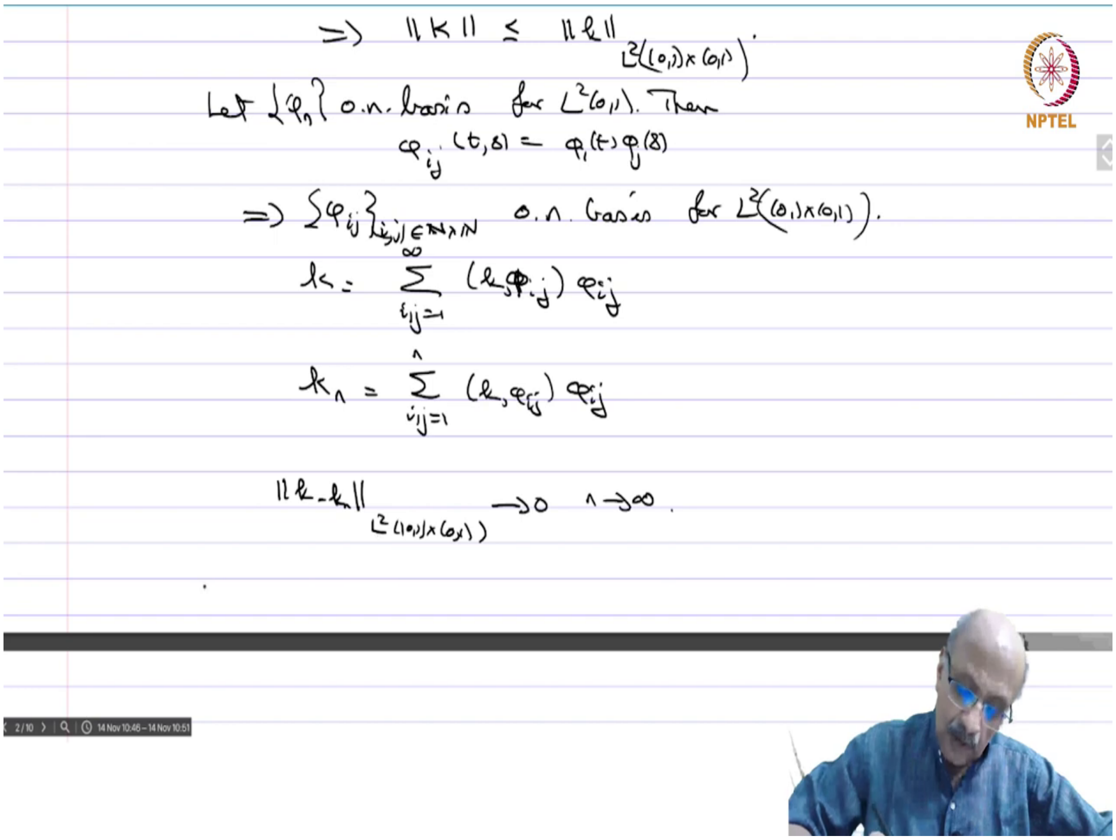

Example. Let $k \in L^2((0,\ 1) \times\ (0,\ 1))$. Let $f \in L^2(0,\ 1)$.Define $K(f)(t) = \int_0^1 k(t,\ s) f(s)\, ds$ this is like a similar operator we defined in the $C(0,\ 1)$ now we are defining in L^2. So, now we have $\int_0^1 |K(f)(t)|^2 dt = \int_0^1 \left| \int_0^1 k(t,\ s) f(s)\, ds \right|^2 dt \le \int_0^1 \left(\int_0^1 |k(t,\ s)|^2 ds \right) \left(\int_0^1 |f(s)|^2\, ds \right) dt$

$$= ||f(s)||_2^2 \int_0^1 \int_0^1 |k(t,\ s)|^2 ds\, dt < \infty$$

And this implies that $K(f) \in L^2(0,\ 1)$. And in fact $||K f||_2 \le ||k||_2\, ||f||_2$. Therefore $||K|| \le ||k||_{L^2((0,\ 1) \times\ (0,\ 1))}$. So, that is the norm here. Let $\{\phi_n\}$ be an orthonormal basis for $L^2(0,\ 1)$. Then, you define $\phi_{ij}(t,\ s) = \phi_i(t)\ \phi_j(s)$. And then you have (we have seen this is an exercise) this implies $\{\phi_{ij}\}_{i,j \in N \times N}$ is an orthonormal basis for $L^2((0, 1) \times (0, 1))$. We have done this exercise already. So, we can write $k = \sum_{i+j=1}^{\infty} (k, \phi_{ij}) \phi_{ij}$. So, now you say,

$k_n = \sum_{i+j=1}^{n} (k, \phi_{ij}) \phi_{ij}$. Then $|| k - k_n ||_{L^2((0,1)\times(0,1))} \to 0 \; as \; n \to \infty$. So, now define

$$K_n(f)(t) = \int_0^1 k_n(t, s) f(s)\, ds = \sum_{i=1}^{n} \int_0^1 \sum_{j=1}^{n} (k, \phi_{ij}) \, \phi_j(s) \, f(s) \, ds \;\; \phi_i(t) \, dt$$

So, $K_n : L^2(0, 1) \to L^2(0, 1)$ as we have already seen and

$||K - K_n|| \le ||k - k_n||_{L^2((0,1)\times(0,1))} \to 0 \; as \; n \to \infty$. $K_n f \in span\{\phi_1, . . . , \phi_n\}$. This implies that K_n is of finite rank. So, K is the limit of operators of finite rank and this implies that K is compact and K is called a Hilbert-Schmidt operator.

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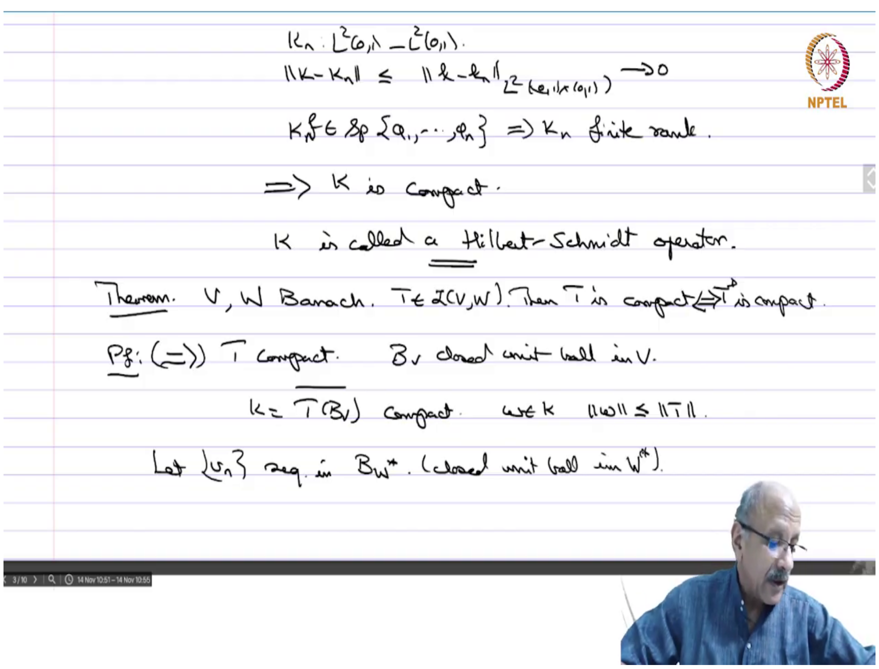

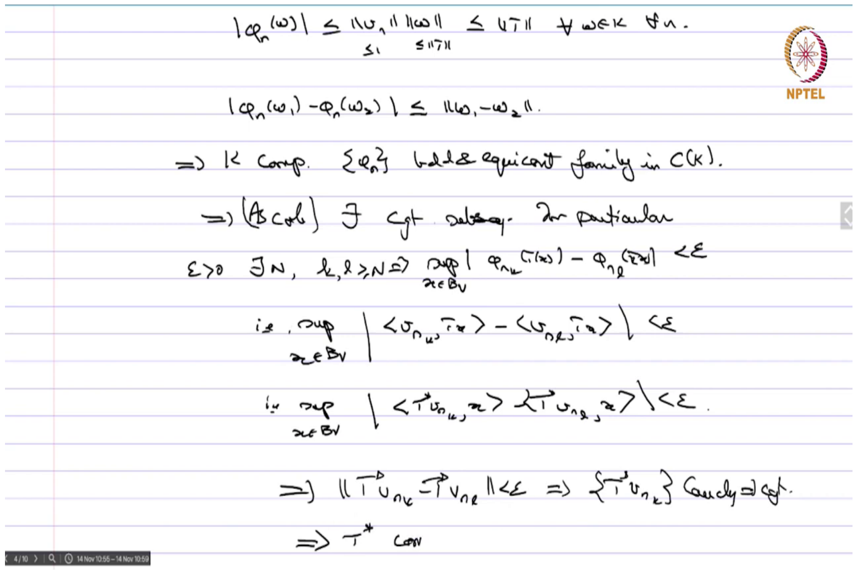

Theorem. $V, \ W$ Banach $T \ \in L(V, \ W)$. Then T is compact if and only if T^* is compact.

Proof. Let us start with T compact. We want to show that T^* is compact. So, B_V is closed unit ball in V and then $K = \overline{T(B_V)}$ is compact. And then if $w \ \in K$ it is limit of elements of the form $T(B_V)$ and therefore $||w|| \leq ||T||$. Let $\{v_n\}$ be a sequence in B_{W^*} that is the closed unit ball in W^*. Define $\phi_n(w) = v_n(w) =< v_n, w>_{W^*, W}, \ v_n \in W^*, \ \forall w \in \ K$.

Then $|| \phi_n(w)|| \ \leq \ ||v_n|| \, ||w||$, this is $||v_n|| \ \leq 1$ and $||w|| \leq \ ||T|| \ \ \forall w \in \ K$ and $\forall n$. And $|\phi_n(w_1) - \phi_n(w_2)| \leq \ ||w_1 - w_2|| \ \Rightarrow K$ compact and $\{\phi_n\}$ is bounded and equi-continuous family in $C(K)$. By Ascoli there exists a convergent subsequence. So, convergent subsequence means also Cauchy, so in particular given any $\epsilon > 0$, $\exists N$ such that $k, \ l \geq N$ implies $\sup |\phi_{n_k}(Tx) - \phi_{n_l}(T x)| < \ \epsilon, \ \ \forall x \in B$. But what is that? That is

$\sup_{x \in B_V} |(v_{n_k}, \ Tx) - (v_{n_l}, \ Tx)| < \epsilon$ i.e. $\sup_{x \in B_V} |(T^* v_{n_k}, \ x) - (T^* v_{n_l}, \ x)| < \epsilon$

$\Rightarrow \ ||T^{*}v_{n_k} - T^{*}v_{n_l}|| < \epsilon \ \Rightarrow \{T^{*}v_{n_k}\}$ is Cauchy $\Rightarrow$ convergent and therefore this tells you therefore for every $v_n \in T(W^{*})$, there is a subsequence which such that $\{T^{*}v_{n_k}\}$ is convergent. Therefore, T^{*} is compact.

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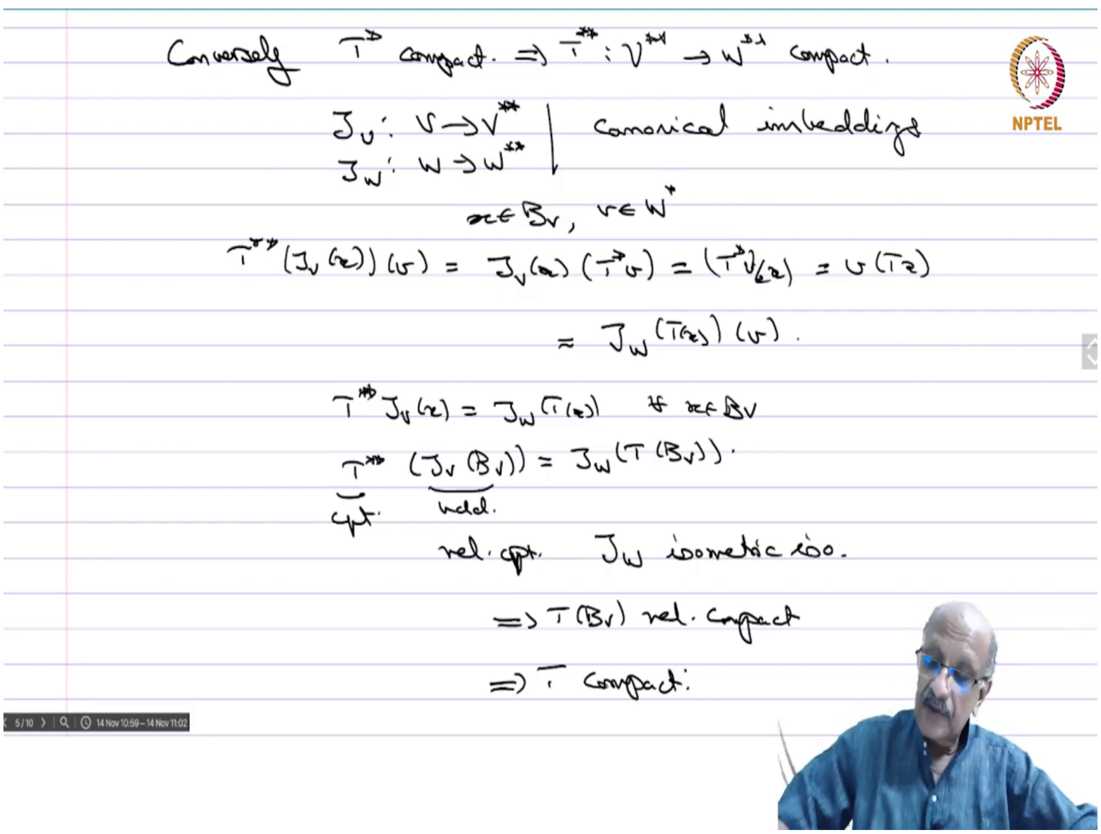

Conversely, T^{*} is compact. So, this implies $T^{**}: V^{**} \rightarrow W^{**}$ is compact. So, then you take $J_V: V \rightarrow V^{**}$ and $J_W: W \rightarrow W^{**}$ canonical embeddings. $x \in B_V$, $v \in W^{*}$.

$$T^{**}(J_V(x))(v) = J_V(x)\, T^{*}(v) = (T^{*}v)x = v(T\,x) = J_W(T(x))(v).$$

So, $T^{**}J_V(x) = J_W(T(x)) \quad \forall\, x \in B_V$. And therefore we have $T^{**}J_V(B_V) = J_W(T(B_V))$.

But B_V is bounded set, $J_V(B_V)$ is bounded, and T^{**} is compact and $J_W(T(B_V))$ is relatively compact. J_W is isometric isomorphism and therefore this implies $T(B_V)$ is relatively compact and that means T is compact and that completes the proof of this theorem.