

Functional Analysis
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Lecture No. 64
Compact operators – Part 2

We saw the limit of compact operators is compact, and since every operator with finite rank is compact, the limit of operators with a finite rank is compact. Now, the question we wanted to ask “is the limit of any compact operator of finite rank?” That is not generally true. So, but here is a situation, where it is true:

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Prop Let V be Banach and W a Hilbert sp. Let $T \in \mathcal{K}(V, W)$. Then T is the limit of operators of finite rank.

Pf: B closed unit ball in V . $K = \overline{T(B)}$ compact

$\epsilon > 0$. Let $f \in W$, $i \in I$ finite indexing set s.t.

$$K \subset \bigcup_{i \in I} B_W(f_i, \epsilon)$$

Let $G = \text{span} \{f_i : i \in I\} \Rightarrow \dim(G) < +\infty$. $P: W \rightarrow G$ orthog. proj.
 $\Rightarrow G$ closed

Then $P \circ T$ is of finite rank

$x \in B \exists i_0 \in I$ s.t. $\|Tx - f_{i_0}\| < \epsilon$. $f_{i_0} \in G, Pf_{i_0} = f_{i_0}$
 $\|P\| = 1$

$$\|P \circ T(x) - f_{i_0}\| = \|P(T(x) - f_{i_0})\| < \epsilon$$

$$\Rightarrow \|P \circ T(x) - T(x)\| < 2\epsilon \quad \forall x \in B \Rightarrow \|P \circ T - T\| < 2\epsilon$$

Proposition. Let V be Banach and W be a Hilbert space. Let $T \in \mathcal{K}(V, W)$ then T is the limit of operators of finite rank.

Proof: B closed unit ball in V and $K = \overline{T(B)}$ is compact. Let $\epsilon > 0$ and let $f \in W$, I finite indexing set, such that $K \subset \bigcup_{i \in I} B_W(f_i, \epsilon)$. Let $G = \text{span} \{f_i / i \in I\}$. We do not know about its linear independence, but this implies that $\dim(G)$ is finite. Let $P: W \rightarrow G$ be the orthogonal

projection. So, $\dim(G) < \infty \Rightarrow G$ is closed and $P: W \rightarrow G$ is the orthogonal projection. Then $P \circ T$ is of finite rank. Because T takes V to W and P takes W to G . G is a finite dimension space therefore $P \circ T$ is of finite rank. Now, let $x \in B$ then there exists $i_0 \in I$ such that $\|Tx - f_{i_0}\| \leq \epsilon$. Then $\|P \circ T(x) - f_{i_0}\| = \|P \circ T(x) - Pf_{i_0}\|$ as $f_{i_0} \in G$, therefore $Pf_{i_0} = f_{i_0}$. So, P being an orthogonal projection $\|P\|$ is equal to 1. And therefore $\|P \circ Tx - f_{i_0}\| = \|P \circ Tx - Pf_{i_0}\| < \epsilon$ because $\|P\| = 1$. Normally a projection will have norm ≥ 1 . And here we have an orthogonal projection. Therefore it is a unit. So this implies that $\|P \circ T(x) - T(x)\| < 2\epsilon \forall x \in B$ and this implies that $\|P \circ T - T\| < 2\epsilon$. So, this completes the proof.

So, we can approximate every compact operator for any given ϵ by an operator of finite rank. And therefore you have that it is the limit of operators of finite rank.

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Ex: Let $k \in L^2((0,1) \times (0,1))$. Let $f \in L^2(0,1)$.

Define $K(f)(t) = \int_0^1 k(t,s)f(s)ds$.

$$\int_0^1 |K(f)(t)|^2 dt = \int_0^1 \left| \int_0^1 k(t,s)f(s)ds \right|^2 dt$$


$$\leq \int_0^1 \left(\int_0^1 |k(t,s)|^2 ds \right) \left(\int_0^1 |f(s)|^2 ds \right) dt$$

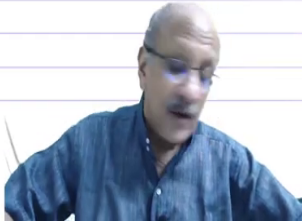
$$= \|f\|_2^2 \int_0^1 \int_0^1 |k(t,s)|^2 ds dt < +\infty$$

$\Rightarrow Kf \in L^2(0,1) \quad \|Kf\|_2 \leq \|k\|_2 \|f\|_2$.

$\Rightarrow \|K\| \leq \|k\|_{L^2((0,1) \times (0,1))}$.

Let $\{q_n\}$ o.n. basis for $L^2(0,1)$. Then $\varphi_{ij}(t,s) = q_i(t)q_j(s)$





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$$\Rightarrow \|K\| \leq \|k\|_{L^2((0,1) \times (0,1))}$$

Let $\{\phi_n\}$ o.n. basis for $L^2(0,1)$. Then
 $\phi_{ij}(t,s) = \phi_i(t)\phi_j(s)$

$\Rightarrow \{\phi_{ij}\}_{i,j \in \mathbb{N}}$ o.n. basis for $L^2((0,1) \times (0,1))$.

$$k = \sum_{i,j=1}^{\infty} (k, \phi_{ij}) \phi_{ij}$$

$$k_n = \sum_{i,j=1}^n (k, \phi_{ij}) \phi_{ij}$$

$$\|k - k_n\|_{L^2((0,1) \times (0,1))} \rightarrow 0 \quad n \rightarrow \infty$$



Example. Let $k \in L^2((0, 1) \times (0, 1))$. Let $f \in L^2(0, 1)$. Define $K(f)(t) = \int_0^1 k(t, s)f(s) ds$

this is like a similar operator we defined in the $C(0, 1)$ now we are defining in L^2 . So, now we

$$\begin{aligned} \int_0^1 |K(f)(t)|^2 dt &= \int_0^1 \left| \int_0^1 k(t, s)f(s) ds \right|^2 dt \leq \int_0^1 \left(\int_0^1 |k(t, s)|^2 ds \right) \left(\int_0^1 |f(s)|^2 ds \right) dt \\ &= \|f\|_2^2 \int_0^1 \int_0^1 |k(t, s)|^2 ds dt < \infty \end{aligned}$$

And this implies that $K(f) \in L^2(0, 1)$. And in fact $\|Kf\|_2 \leq \|k\|_2 \|f\|_2$. Therefore

$\|K\| \leq \|k\|_{L^2((0,1) \times (0,1))}$. So, that is the norm here. Let $\{\phi_n\}$ be an orthonormal basis for

$L^2(0, 1)$. Then, you define $\phi_{ij}(t, s) = \phi_i(t)\phi_j(s)$. And then you have (we have seen this is an

exercise) this implies $\{\phi_{ij}\}_{i,j \in \mathbb{N} \times \mathbb{N}}$ is an orthonormal basis for $L^2((0, 1) \times (0, 1))$. We have

done this exercise already. So, we can write $k = \sum_{i+j=1}^{\infty} (k, \phi_{ij}) \phi_{ij}$. So, now you say,

$k_n = \sum_{i+j=1}^n (k, \phi_{ij}) \phi_{ij}$. Then $\|k - k_n\|_{L^2((0,1) \times (0,1))} \rightarrow 0$ as $n \rightarrow \infty$. So, now define

$$K_n(f)(t) = \int_0^1 k_n(t, s) f(s) ds = \sum_{i=1}^n \int_0^1 \sum_{j=1}^n (k, \phi_{ij}) \phi_j(s) f(s) ds \phi_i(t) dt$$


So, $K_n: L^2(0, 1) \rightarrow L^2(0, 1)$ as we have already seen and

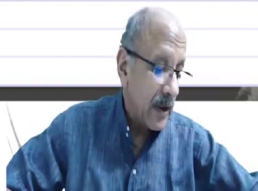
$\|K - K_n\| \leq \|k - k_n\|_{L^2((0,1) \times (0,1))} \rightarrow 0$ as $n \rightarrow \infty$. $K_n f \in \text{span}\{\phi_1, \dots, \phi_n\}$. This implies that K_n is of finite rank. So, K is the limit of operators of finite rank and this implies that

K is compact and K is called a Hilbert-Schmidt operator.

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$k_n \in L^2(0,1) \times L^2(0,1)$
 $\|K - K_n\| \leq \|k - k_n\|_{L^2((0,1) \times (0,1))} \rightarrow 0$
 $K_n f \in \text{Sp}\{\phi_1, \dots, \phi_n\} \Rightarrow K_n$ finite rank.
 $\Rightarrow K$ is compact.
 K is called a Hilbert-Schmidt operator.
Theorem. V, W Banach. $T \in \mathcal{L}(V, W)$. Then T is compact $\Leftrightarrow T^*$ is compact.
Pf: (\Rightarrow) T compact. B_V closed unit ball in V .
 $K = T(B_V)$ compact. $w \in K$ $\|w\| \leq \|T\|$.
 Let $\{v_n\}$ seq. in B_{W^*} (closed unit ball in W^*).


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$| \phi_n(w) | \leq \|v_n\| \|w\| \leq \|T\| \|w\| \quad \forall w \in K \quad \forall n.$
 $| \phi_n(w_1) - \phi_n(w_2) | \leq \|w_1 - w_2\|.$
 $\Rightarrow K$ comp. $\{ \phi_n \}$ bounded equicont family in $C(K)$.
 \Rightarrow (Ascoli) \exists cgt subseq. in particular
 $\epsilon > 0 \exists N, k, l \geq N \Rightarrow \sup_{x \in B_V} | \phi_{n_k}(Tx) - \phi_{n_l}(Tx) | < \epsilon$
i.e. $\sup_{x \in B_V} | \langle v_{n_k}, Tx \rangle - \langle v_{n_l}, Tx \rangle | < \epsilon$
i.e. $\sup_{x \in B_V} | \langle T v_{n_k}, x \rangle - \langle T v_{n_l}, x \rangle | < \epsilon.$
 $\Rightarrow \| T v_{n_k} - T v_{n_l} \| < \epsilon \Rightarrow \{ T v_{n_k} \}$ Cauchy \Rightarrow cgt.
 $\Rightarrow T^*$ comp

Theorem. V, W Banach $T \in L(V, W)$. Then T is compact if and only if T^* is compact.

Proof. Let us start with T compact. We want to show that T^* is compact. So, B_V is closed unit ball in V and then $K = \overline{T(B_V)}$ is compact. And then if $w \in K$ it is limit of elements of the form $T(B_V)$ and therefore $\|w\| \leq \|T\|$. Let $\{v_n\}$ be a sequence in B_{W^*} that is the closed unit ball in W^* . Define $\phi_n(w) = v_n(w) = \langle v_n, w \rangle_{W^*, W}$, $v_n \in W^*$, $\forall w \in K$.

Then $\| \phi_n(w) \| \leq \|v_n\| \|w\|$, this is $\|v_n\| \leq 1$ and $\|w\| \leq \|T\| \quad \forall w \in K$ and $\forall n$. And $| \phi_n(w_1) - \phi_n(w_2) | \leq \|w_1 - w_2\| \Rightarrow K$ compact and $\{ \phi_n \}$ is bounded and equi-continuous family in $C(K)$. By Ascoli there exists a convergent subsequence. So, convergent subsequence means also Cauchy, so in particular given any $\epsilon > 0$, $\exists N$ such that $k, l \geq N$ implies $\sup | \phi_{n_k}(Tx) - \phi_{n_l}(Tx) | < \epsilon, \quad \forall x \in B$. But what is that? That is

$$\sup_{x \in B_V} | (v_{n_k}, Tx) - (v_{n_l}, Tx) | < \epsilon \text{ i.e. } \sup_{x \in B_V} | (T^* v_{n_k}, x) - (T^* v_{n_l}, x) | < \epsilon$$

$\Rightarrow \|T^* v_{n_k} - T^* v_{n_l}\| < \epsilon \Rightarrow \{T^* v_{n_k}\}$ is Cauchy \Rightarrow convergent and therefore this tells you

therefore for every $v_n \in T(W^*)$, there is a subsequence which such that $\{T^* v_{n_k}\}$ is convergent.

Therefore, T^* is compact.

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Conversely T compact $\Rightarrow T^{**}: V^{**} \rightarrow W^{**}$ compact.

$J_V: V \rightarrow V^{**}$
 $J_W: W \rightarrow W^{**}$

Canonical embeddings
 $x \in B_V, v \in W^*$

$$T^{**}(J_V(x))(v) = J_V(x)(T^*v) = (T^*v)(x) = v(Tx) = J_W(Tx)(v)$$

$$= J_W(Tx)(v)$$

$$T^{**}J_V(x) = J_W(Tx) \quad \forall x \in B_V$$

$$\underbrace{T^{**}}_{\text{cpt.}}(\underbrace{J_V(B_V)}_{\text{weak.}}) = \underbrace{J_W(T(B_V))}_{\text{rel. cpt.}}$$

J_W isometric iso.

$\Rightarrow T(B_V)$ rel. compact

$\Rightarrow T$ compact.

Conversely, T^* is compact. So, this implies $T^{**}: V^{**} \rightarrow W^{**}$ is compact. So, then you take

$J_V: V \rightarrow V^{**}$ and $J_W: W \rightarrow W^{**}$ canonical embeddings. $x \in B_V, v \in W^*$.

$$T^{**}(J_V(x))(v) = J_V(x)(T^*v) = (T^*v)(x) = v(Tx) = J_W(Tx)(v).$$

So, $T^{**}J_V(x) = J_W(Tx) \quad \forall x \in B_V$. And therefore we have $T^{**}J_V(B_V) = J_W(T(B_V))$.

But B_V is bounded set, $J_V(B_V)$ is bounded, and T^{**} is compact and $J_W(T(B_V))$ is relatively compact. J_W is isometric isomorphism and therefore this implies $T(B_V)$ is relatively compact and that means T is compact and that completes the proof of this theorem.