Functional Analysis
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Lecture No. 63
Compact operators – Part 1

We begin the last chapter of this course. We are going to study compact operators, a special class of continuous linear operators which mimic a lot of properties of finite dimensional operators. So, we start with the definition.

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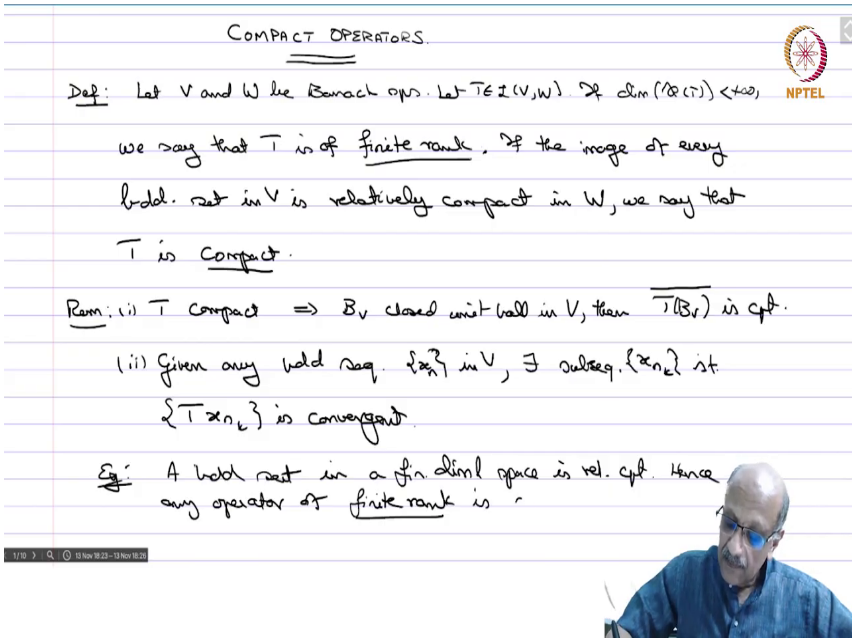

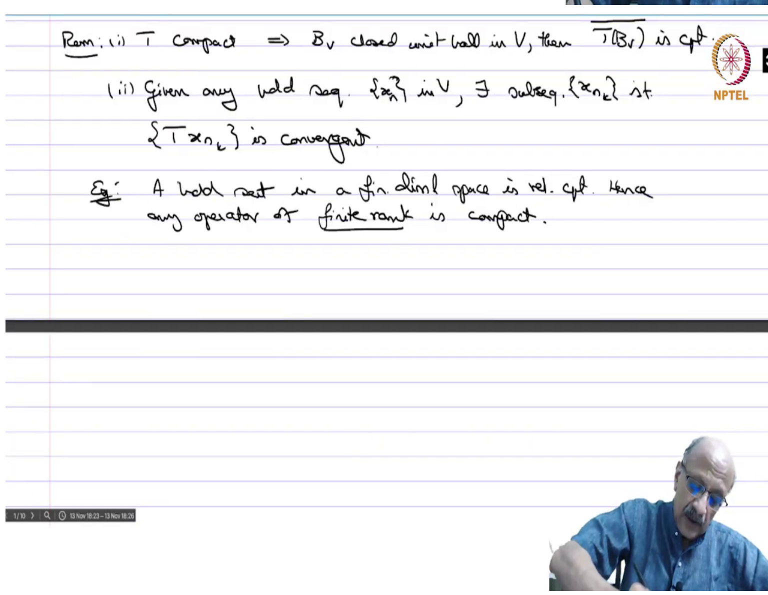

Definition: Let V and W be banach spaces. Let $T \in L(V, W)$. If the dimension of the range of T is finite. We say that T is of finite rank. If the image of every bounded set in V is relatively compact in W, we say that T is compact.

Remark: (i) T is compact ⇒ if B_V is the closed unit ball in V then $T(B_V)$ need not be closed

$\overline{T(B_V)}$ is compact.

(ii) Given any bounded sequence $\{x_n\}$ in V, there exists a subsequence $\left\{x_{n_k}\right\}$ such that $\left\{T\left(x_{n_k}\right)\right\}$ is convergent.

Because $\{x_n\}$ is in the bounded set, its image is relatively compact and in Matrix space, compact means sequentially compact and therefore you have a convergent subsequence.

Examples. Any bounded set in a finite dimensional space is relatively compact, hence any operator of finite rank is compact.

Because the image will go into a finite dimensional space and it is bounded, so it will be relatively compact, if V is finite dimensional, then T: $V \to W$ is always compact. Because it is finite rank and therefore it is always compact.

Example. V is infinite dimensional. I: $V \to V$ is not compact because B_V is not compact.

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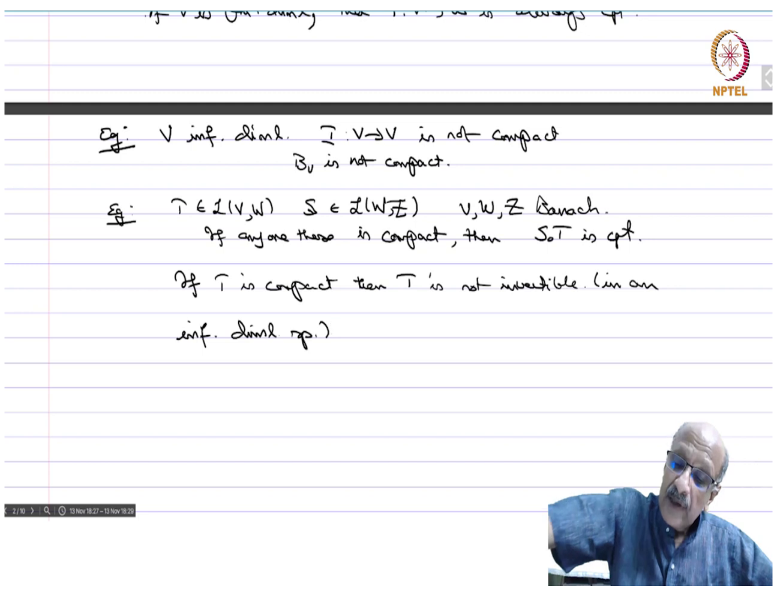

Example. $T \in L(V, W)$ and $S \in L(W, Z)$, V, W, Z are banach. If any one of these is compact, then $S \circ T$ is compact. Because continuous linear maps take bounded sets to bounded sets and compact sets to compact sets.

When you have any bounded set, then the composition will always be a relatively compact set. So, if T is compact then T is not invertible. Because if it were invertible, then $T \circ T^{-1}$ or $T^{-1} \circ T$ is the identity map. So, if one of them is compact, then the identity must be compact, which is not true in an infinite dimensional space. So, up to now, we have been sort of not giving very serious examples except the identity is not compact. That depends on the deep theorem.

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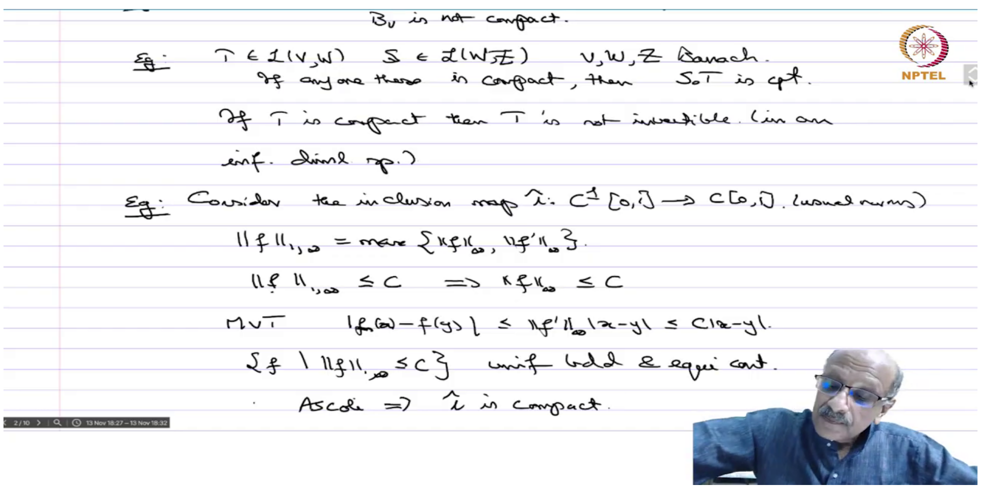

Example. consider the inclusion map $\hat{i} : C^1[0, 1] \to C[0, 1]$. Now, C^1 has the norm namely

$||f||_{1,\infty} = \max\{||f||_\infty, ||f'||_\infty\}$. The usual norms make them Banach spaces. So, with these usual norms we want to show that the inclusion map is compact.

$||f||_{1,\infty} \leq C \Rightarrow ||f||_\infty \leq C$. Now, we also have that $||f'||_\infty \leq C$.

By the mean value theorem, $|f(x) - f(y)| \leq ||f'||_\infty |x - y| \leq C|x - y|$.

Therefore, $\{f \,/\, ||f'||_{\infty} \leq C\}$ is uniformly bounded and equi-continuous and therefore by Ascoli's theorem, the image is relatively compact because $[0,\ 1]$ is a compact set and therefore implies $\hat{i}$ is compact. So, this is just a direct consequence of Ascoli's theorem.

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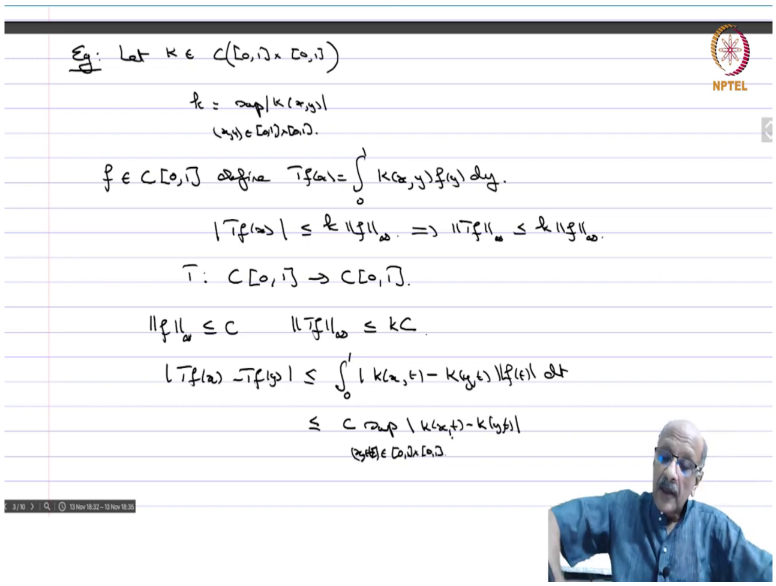

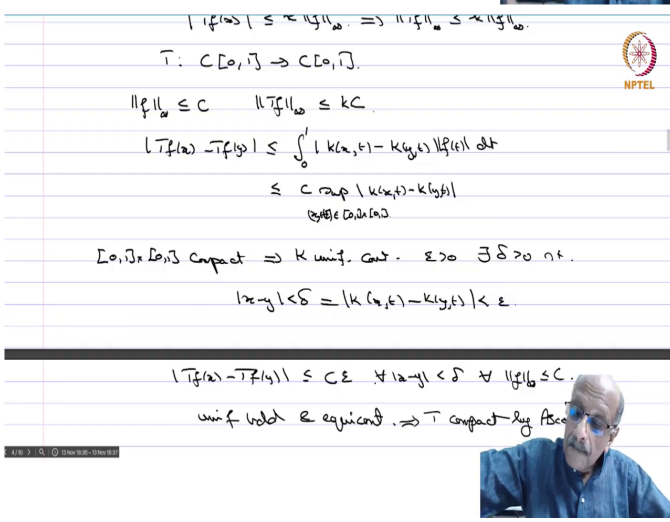

Example. Let $K \in C([0,\ 1] \times [0,\ 1])$, $k = \sup_{x,\ y \in [0,\ 1]\times[0,\ 1]} |K(x,\ y)|$. For $f \in C[0,\ 1]$, define $Tf(x) = \int_0^1 K(x,\ y)\, f(y)\, dy$. $K(x,\ y)$, $f(y)$ are all continuous functions on the finite intervals. So, they are all integrable and you have $|Tf(x)| \leq k\, ||f||_{\infty}$ and therefore,

$||Tf||_\infty \leq k||f||_\infty$. And therefore, T is a continuous linear transformation of $C[0, 1]$ and Tf is also continuous, you can check.

So, $T: C[0, 1] \to C[0, 1]$. You have to check that this is a continuous map. Because of uniform continuity of K. If $||f||_\infty \leq C$, then $||Tf||_\infty \leq kC$ and therefore they are all uniformly bounded.

$$|Tf(x) - Tf(y)| \leq \int_0^1 |K(x, t) - K(y, t)|\, |f(t)|\, dt \leq C \sup_{x, y, t \in [0, 1]\times[0,1]} |K(x, t) - K(y, t)|.$$

$[0, 1] \times [0, 1]$ is compact. Therefore, K is uniformly continuous, given any $\epsilon > 0$, $\exists\, \delta > 0$, such that $|x - y| < \delta \Rightarrow |K(x, t) - K(y, t)| < \epsilon$.

Therefore $|Tf(x) - Tf(y)| < C\epsilon$, $\forall\, |x - y| < \delta$ and $\forall ||f||_\infty \leq C$.

And therefore, again uniformly bounded and equi continuous $\Rightarrow$ T is compact by Ascoli. So, Ascoli's theorem again tells you that T is compact.

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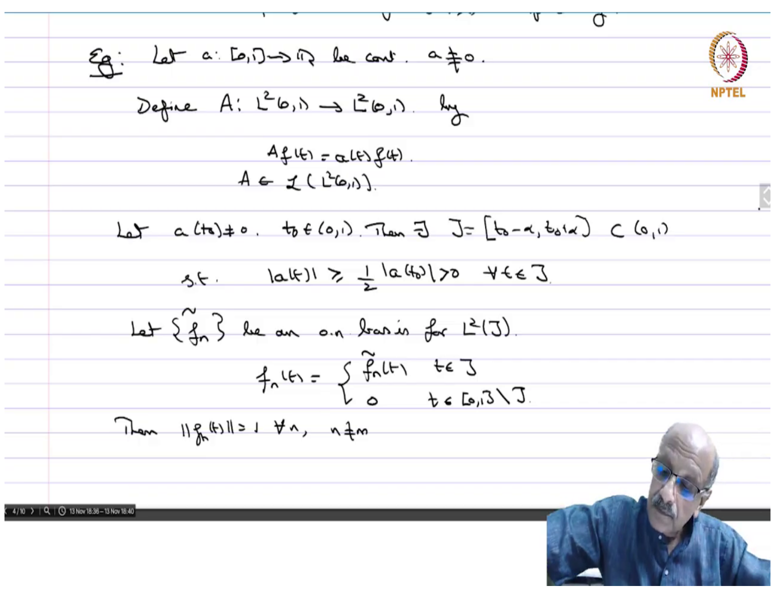

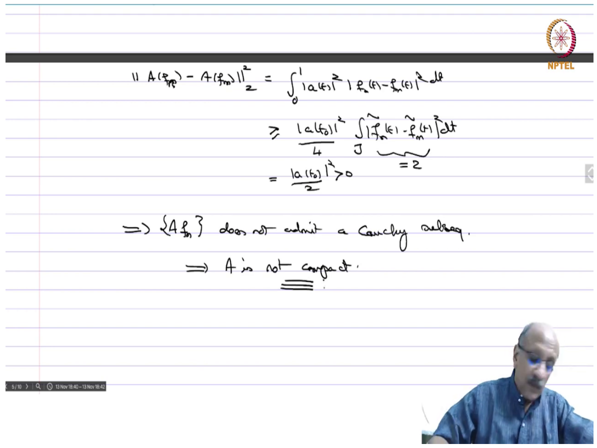

Example. Let $a\colon [0, 1] \to \mathbb{R}$ be continuous and a is not identically 0.

Define $A : L^2(0, 1) \to L^2(0, 1)$ by $Af(t) = a(t)\, f(t)$. So, you are just multiplying continuous functions. A continuous function on a compact set is bounded. So, if you are multiplying by a bounded function and L^2 function it is again L^2 function so A takes L^2 into L^2. Also you can check that $A \in L(L^2(0, 1))$. You have to do is $||Af(t)||^2 = |a(t)|^2 |f(t)|^2 < \sup ||A||^2 ||f||^2$. So, it is also a bounded linear operator. Let $a(t_0) \neq 0$. $t_0 \in (0, 1)$. Then $\exists\, J = [t_0 - \alpha,\, t_0 + \alpha] \subset (0, 1)$, such that $|a(t)| \geq \frac{1}{2}|a(t_0)| > 0 \quad \forall t \in J$. It is just a continuity condition. $a(t_0)$ is non-zero so in a small interval $a(t)$ will also be non-zero. And we can choose the $\epsilon = \frac{1}{2}|a(t_0)|$ and therefore you will get this condition.

Let $\widetilde{f}_n$ be an orthonormal basis for $L^2(J)$. It is also an interval, so you have a separable space so you have an orthonormal basis. So, you know,

$f_n(t) = \widetilde{f}_n(t),\ \ if\ t \in J$ and $f_n(t) = 0,\ \ if\ t \in [0, 1] \setminus J$

Then $||f_n(t)|| = 1 \ \ \forall n.$ If $n \neq m,$ we have

$||A(f_n) - A(f_m)||_2^2 = \int_0^1 |a(t)|^2 |f_n(t) - f_m(t)|^2 \; dt$ and this integral will survive only over J because outside J, $f_n = f_m = 0$.

$$||A(f_n) - A(f_m)||_2^2 = \int_0^1 |a(t)|^2 |f_n(t) - f_m(t)|^2 \; dt \geq \frac{1}{4} |a(t_0)|^2 \int_J |\widetilde{f}_n(t) - \widetilde{f}_m(t)|^2 \, dt$$

$|\widetilde{f}_n(t) - \widetilde{f}_m(t)|^2 = 2$ because $\widetilde{f}_n$ an orthonormal basis.

So, $||A(f_n) - A(f_m)||_2^2 \geq \frac{1}{2} |a(t_0)|^2 > 0$. This means that $\{A f_n\}$ does not admit a Cauchy subsequence. And therefore, this implies A is not compact.

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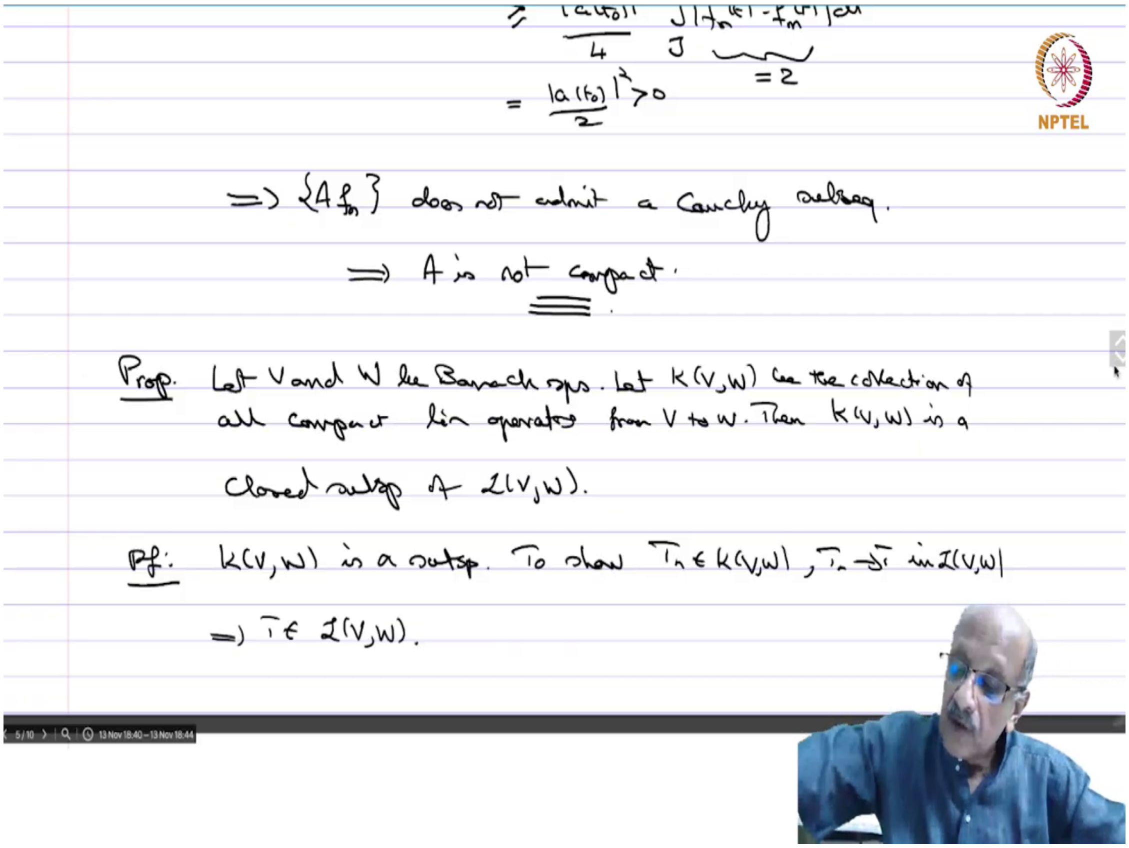

We have given a fairly good number of examples. So, now let us prove the following properties of compact operators.

Proposition. Let V and W be Banach spaces. Let $K(V,\ W)$ be the collection of all compact linear operators from V to W then $K(V,\ W)$ is a closed subspace of $L(V,\ W)$.

Proof. $K(V,\ W)$ is a subspace (if sum of to compact of operators is again compact, scalar multiple is again compact, therefore $K(V,\ W)$ is a subspace). So, we have to show

$T_n \in K(V,\ W), \quad T_n \to T \Rightarrow \quad T \in L(V,\ W)$. So, the limit of compact operators in the operator norm is in fact, a compact operator.

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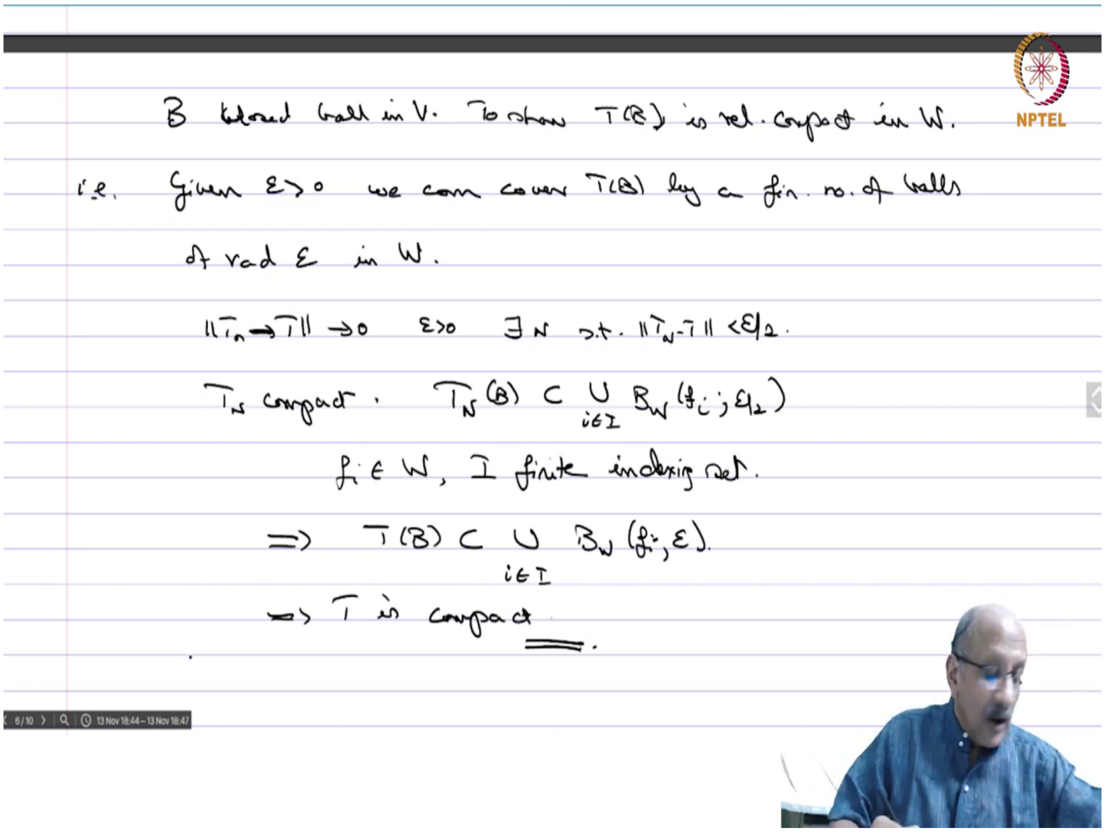

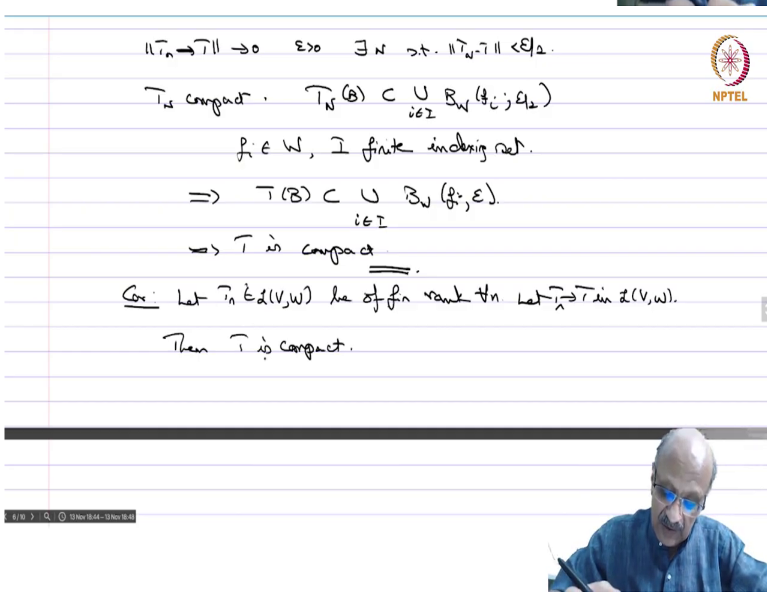

Let B be a closed ball in V. To show $T(B)$ is relatively compact in W. We are in the Metric space and therefore it is enough to show that it is totally bounded. Given ϵ. Then we can cover $T(B)$ by a finite number of balls of radius ϵ in W. This is what we want to show. Now, $||T_n - T|| \to 0$, so given $\epsilon > 0 \quad \exists\, N$ such that $||T_n - T|| < \epsilon/2$.

Given T_n is compact, $T_n(B) \subset \bigcup_{i \in I} B_W(f_i, \epsilon/2)$ where $f_i \in W$ and I is a finite indexing set.

Now, $||T_n - T|| < \epsilon/2$ and therefore this implies that $T(B) \subset \bigcup_{i \in I} B_W(f_i, \epsilon/2)$ (this is immediate to check). And therefore, this implies that T is compact.

Corollary: let $T_n \in L(V, W)$ be compact and of finite rank. Let $T_n \to T$ in $L(V, W)$. Then T is compact. So, that means because any mapping of finite rank is compact. And now the limit of compact operators is compact, that is what we have already seen. So, the limit of operators of finite rank is always a compact operator. And an interesting question is the converse proof i.e. can every compact operator be approximated by operators of finite rank? It is not always true, but we did see cases where it is true then we will also do some exercises in this connection and we continue with this.