Functional Analysis
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The Institute of Mathematical Sciences
Lecture No. 62

Exercises – Part 3

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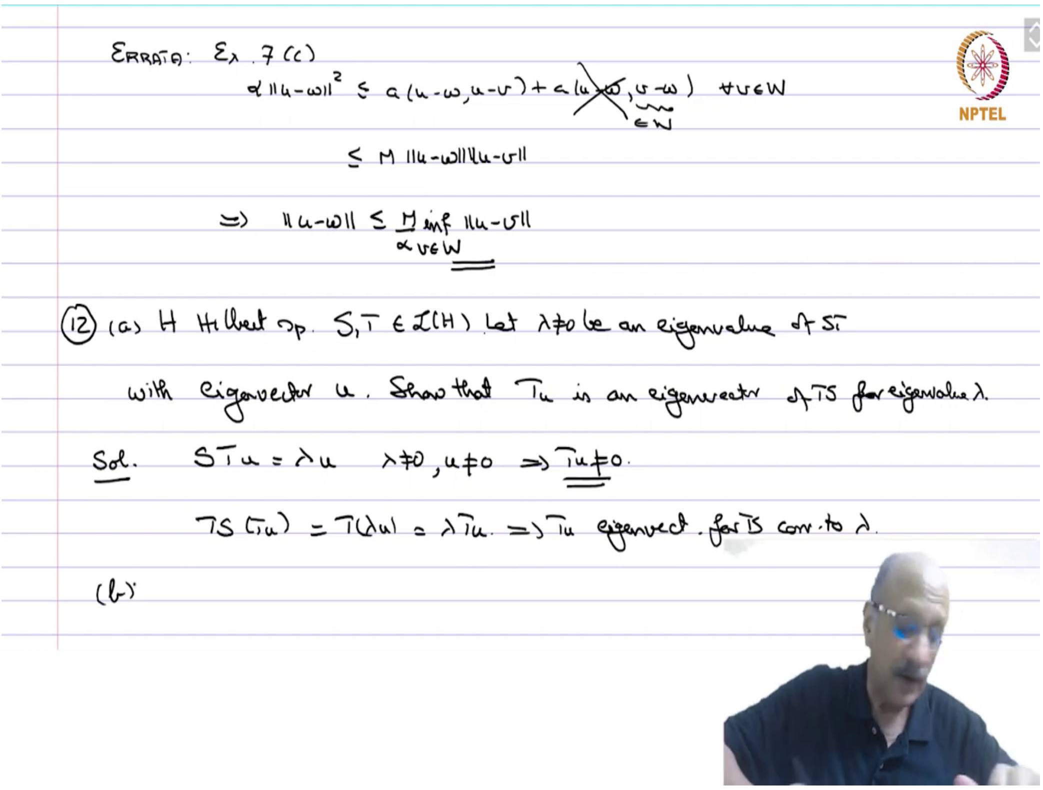

Before we continue with the exercises is a small correction. **Exercise 7c**. We showed that $\alpha||u - w||^2 \leq a(u - w, u - v) + a(u - w, v - w)$ for all $v \in W$ and this since $v - w \in W$, $a(u - w, v - w) = 0$. So, $\alpha||u - w||^2 \leq M||u - w||\,||u - v||$ and therefore, you get that $||u - w|| \leq \frac{M}{\alpha}\inf_{v \in W}||u - v||$.

12. (a) H Hilbert space and $S, T \in L(H)$. Let $\lambda \neq 0$ be an eigenvalue of ST with eigenvector u. Show that Tu is an eigenvector of TS for eigenvalue λ.

Solution. You have $S\,T\,u = S(Tu) = \lambda u$, $\lambda \neq 0$, and obviously $u \neq 0$. So, this implies that Tu cannot be 0. So, $Tu \neq 0$. Now, $TS(Tu) = T(STu) = T(\lambda u) = \lambda T(u)$. Therefore, Tu is an eigenvector of TS for eigenvalue λ. So, that is a very trivial statement.

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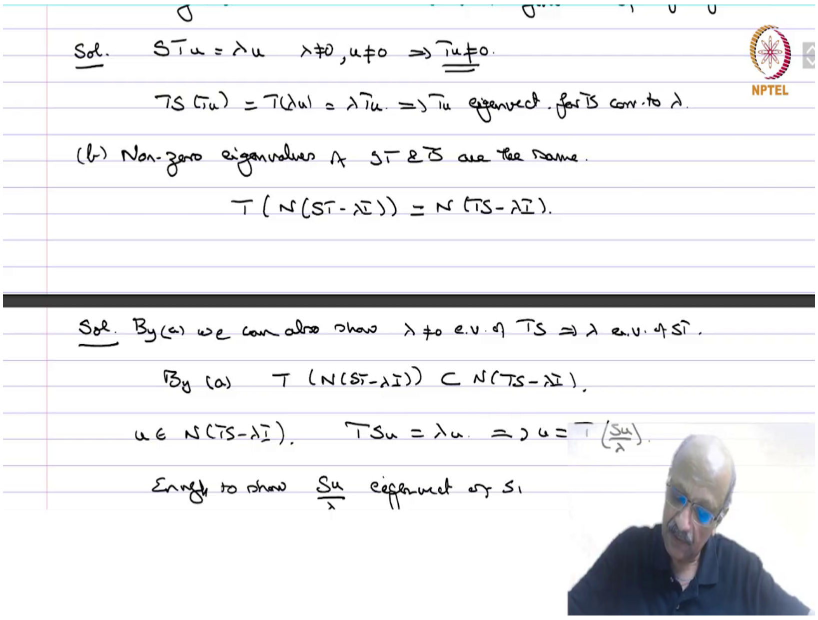

(b) Non-zero eigenvalues of ST and TS are the same and $T(N(ST - \lambda I)) = N(TS - \lambda I)$.

Solution. By (a) we can also show $\lambda \neq 0$ eigenvalue of $TS \Rightarrow \lambda$ eigenvalue of ST. So, that is the same thing as T and S is no nothing sacred about that. By (a) again if u is an eigenvector of ST, then $T\,u$ is eigenvector of TS and therefore, we have

$T\,(N(ST - \lambda\, I)) \subset N(TS - \lambda I)$. Now we have to show the converse. $u \in N(TS - \lambda I)$, we have $TS\,u = \lambda\, u \Rightarrow u = T(\frac{S\,u}{\lambda})$. Now it is enough to show $\frac{S\,u}{\lambda}$ is an eigenvector of ST

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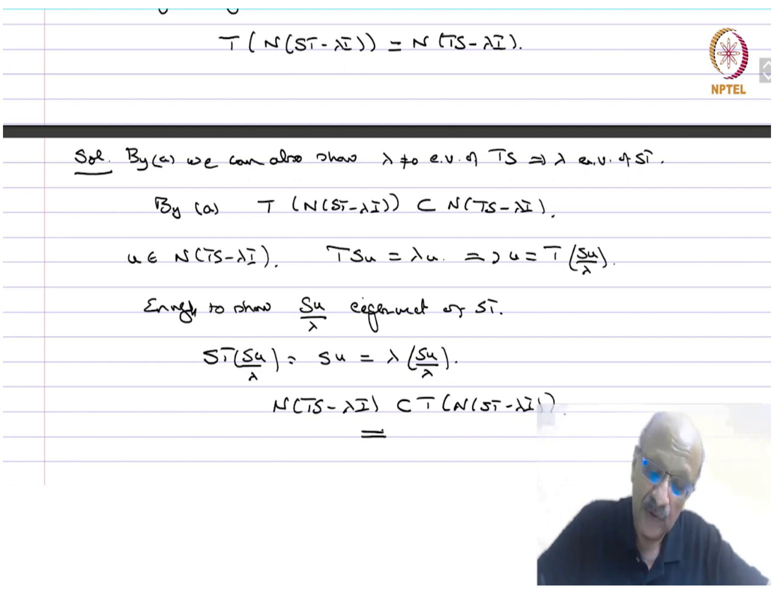

So, we have to show the $ST\left(\frac{Su}{\lambda}\right) = S\left(\frac{TSu}{\lambda}\right) = Su$, that completes the proof. Therefore, we have shown $N(TS - \lambda I) \subset T\,(N(ST - \lambda\, I))$. So, we have shown that all the non-zero eigenvalues of TS and ST are the same.

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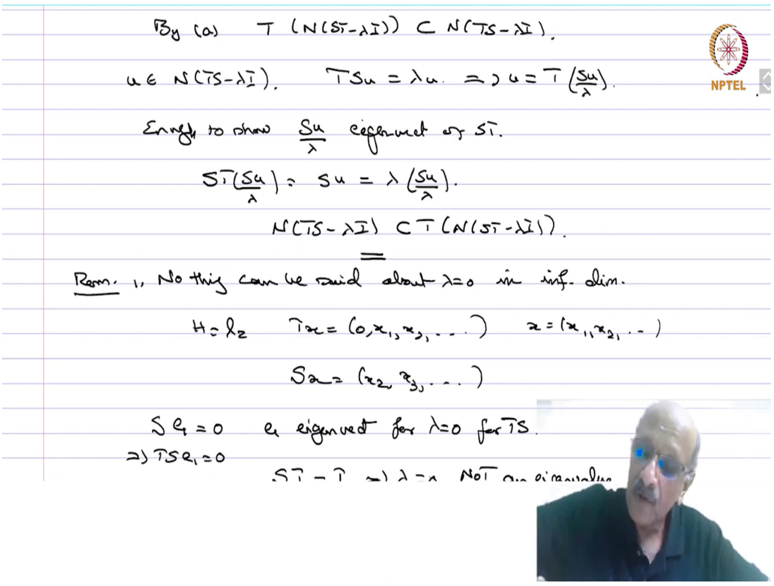

Remark 1: nothing can be said about $\lambda = 0$ in infinite dimensions. For instance you take

$H = l_2$ and $Tx = (0, x_1, x_2, ...)$, $x = (x_1, x_2, ...)$ and $Tx = (x_2, x_3, ...)$. So, both of them are bounded linear operators. If you take e_1, then $S\,e_1 = 0$, so, e_1 is an eigenvector for $\lambda = 0$ for TS, because this also implies $TSe_1 = 0$.

On the other hand if you take ST, this is the identity map, implies $\lambda = 0$ not an eigenvalue.

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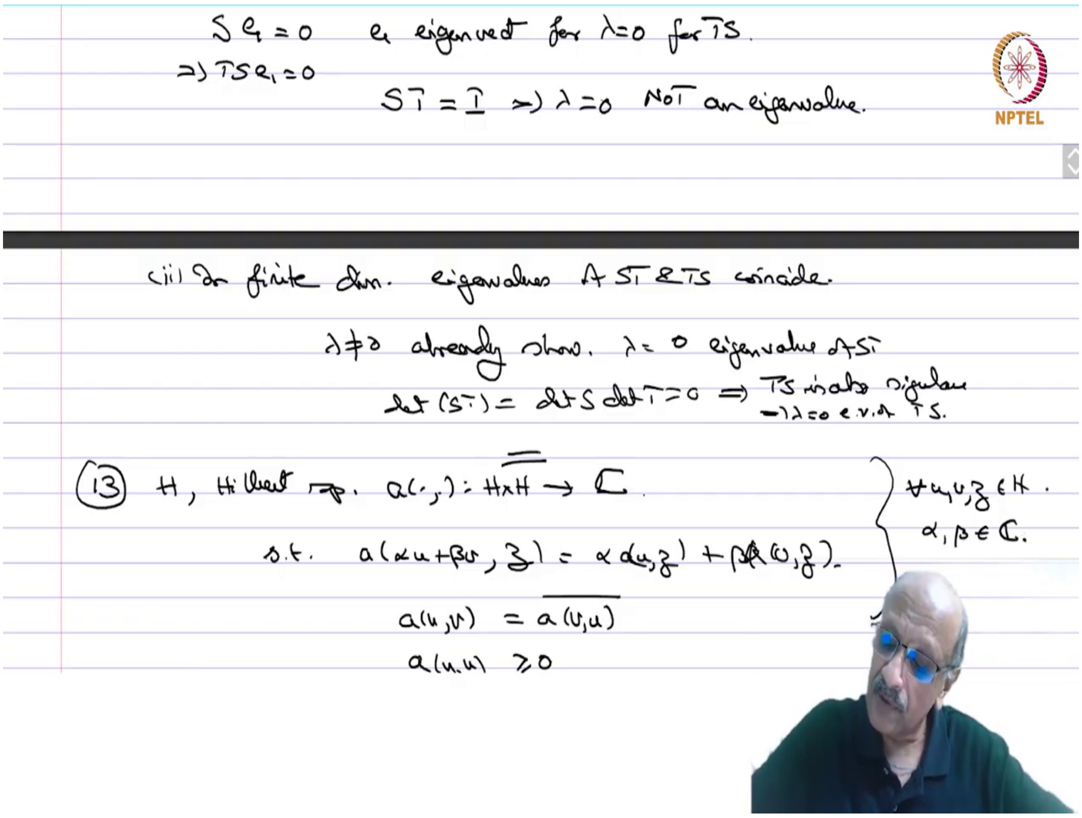

In finite dimensions we have eigenvalues of ST and TS coincide. We have already shown for lambda naught you know already shown. So, if 0 is an eigenvalue of ST, means, $\det ST = \det S \det T = 0 \Rightarrow TS$ is also singular, because its determinant is also 0 implies $\lambda = 0$ is an eigenvalue of TS. In the finite dimensional case in fact you can show the characteristic polynomials are in fact the same. Now, so that you might have done in linear algebra.

13. H, Hilbert space, $a(. \, , \, .)$: $H \times H \rightarrow \mathbb{C}$, such that

$$a(\alpha u + \beta v, z) = \alpha a(u, z) + \beta a(v, z) \quad \forall u, v, z \in H \text{ and } \forall \alpha, \beta \in \mathbb{C}$$

that is linear in the first variable. Then $a(u, v) = \overline{a(u, v)}$, and $a(u, u) \geq 0$. This is almost looking like an inner product except that the third condition is a bit weak.

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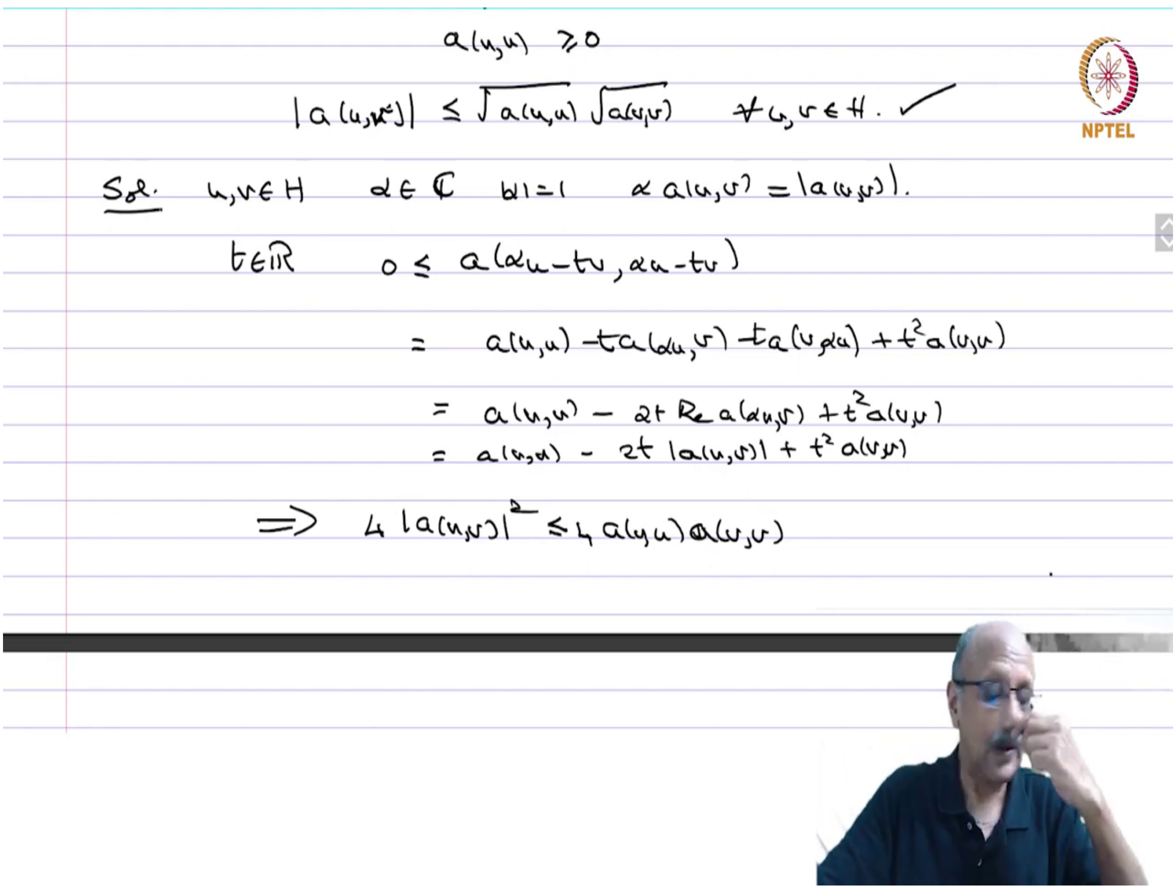

It does not say that this is a norm in fact, $a(u,\ u)\ =\ 0$ may not mean that $u\ =\ 0$. But even then you have the analogue of the Cauchy Schwarz inequality, i.e.,

$|a(u,\ v)|\ \leq\ \sqrt{a(u,\ u)}\sqrt{a(v,\ v)}$, the square root is well defined for all $\forall\, u,\ v\ \in\ H$. So, this proof is identical as in the Cauchy Schwarz inequality.

Solution. You have $u,\ v\ \in\ H$ and then take $\alpha\ \in\ \mathbb{C}$, $|\alpha|\ =\ 1$, and $\alpha\, a(u,\ v)\ =\ |a(v,\ v)|$.

Let $t\ \in\ \mathbb{R}$. So, you have $0\ \leq\ a(\alpha u\ -\ tv,\ \alpha u\ -\ tv)$ by the third property

$$=\ a(u,\ u)\ -\ t\, a(\alpha\, u,\ v)\ -\ t\, a(v,\ \alpha u)\ +\ t^2\, a(v,\ v)$$

Now, by the second property $a(\alpha\, u,\ v)\ =\ \overline{a(v,\ \alpha u)}$. Therefore,

$$a(u,\ u)\ -\ t\, a(\alpha\, u,\ v)\ -\ t\, a(v,\ \alpha u)\ +\ t^2\, a(v,\ v)\ =\ a(u,\ u)\ -\ 2t\, Re\, a(\alpha\, u,\ v)\ +\ t^2\, a(v,\ v)$$

$$=\ a(u,\ u)\ -\ 2t\, |a(v,\ v)|\ +\ t^2\, a(v,\ v)$$

And now, you have a quadratic form which is never changed a sign that means the roots of the quadratic should either coincident or imaginary and therefore,

$4\,|a(v,\ v)|^2\ \leq\ 4\,a(u,\ u)\,a(v,\ v).$

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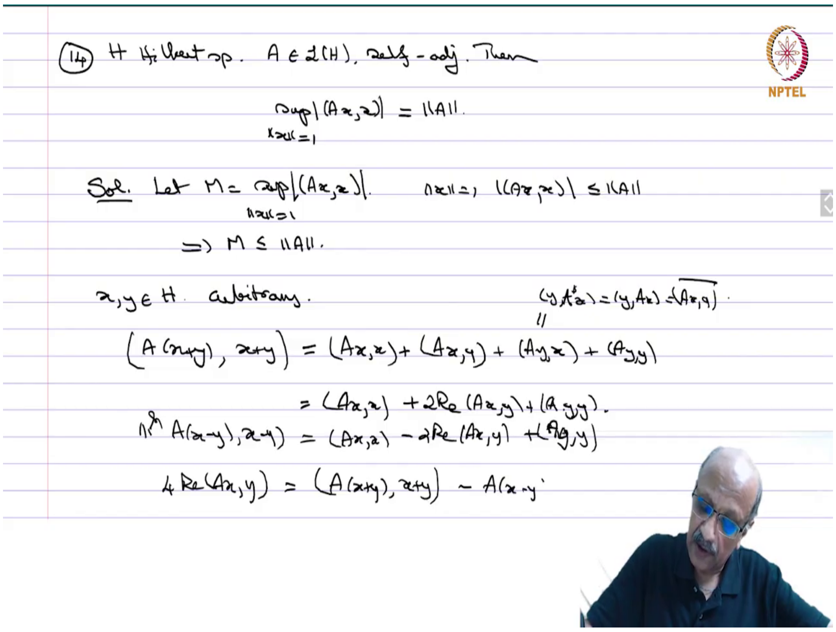

For the Cauchy Schwarz inequality, you do not need the full power of the inner product.

14. H, Hilbert space, $A\ \in\ L(H)$, self-adjoint. Then $\sup_{||x||=1}\ |(Ax,\ x)|\ =\ ||A||$.

So, you have one more formula for the norm.

Solution. Let $M\ =\ \sup_{||x||=1}\ |(Ax,\ x)|$, if $||x||\ =\ 1$, then $|(Ax,\ x)|\ \leq\ ||A||\ ||x||^2\ =\ ||A||$.

Therefore, you have that $M\ \leq\ ||A||$. Now, let $x,\ y\ \in\ H$ be arbitrary. So, let us compute

$(A(x\ +\ y),\ x\ +\ y) = (Ax,\ x)\ +\ (Ax,\ y)\ +\ (Ay,\ x)\ +\ (Ay,\ y)$. Now, what is $(Ay,\ x)$? $(Ay,\ x)\ =\ (y,\ A^*x)\ =\ (y,\ Ax)\ =\overline{(Ax,\ y)}$. And therefore,

$$(A(x\ +\ y),\ x\ +\ y) = (Ax,\ x)\ +\ (Ax,\ y)\ +\ (Ax,\ y)\ +\ (Ay,\ y)$$

$$=\ (Ax,\ x)\ +\ 2\ Re(Ax,\ y)\ +\ (Ay,\ y).$$

Similarly, $(A(x\ -\ y),\ x\ -\ y) = (Ax,\ x)\ -\ 2\ Re(Ax,\ y)\ +\ (Ay,\ y)$

So, when you subtract, you get $4\,Re(Ax,\ y)\ = (A(x\ +\ y),\ x\ +\ y) - (A(x\ -\ y),\ x\ -\ y)$.

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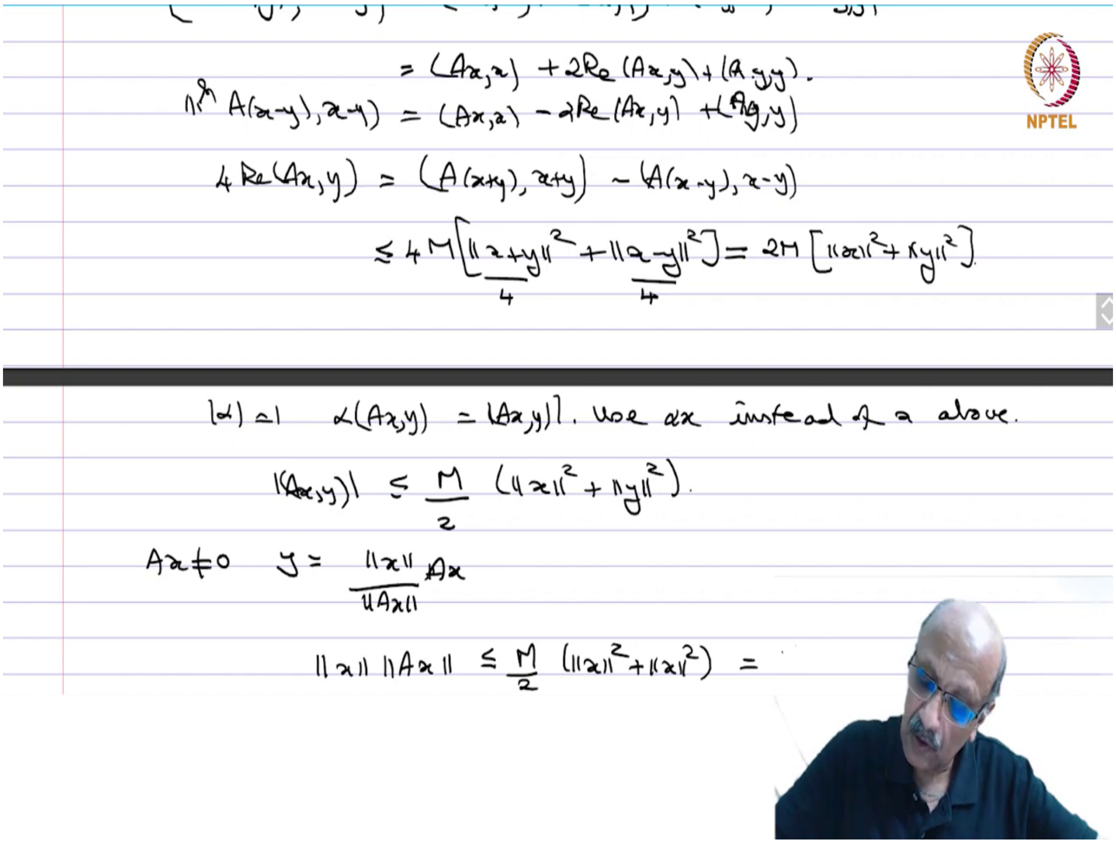

$4\,Re(Ax,\ y)\ = (A(x\ +\ y),\ x\ +\ y) - (A(x\ -\ y),\ x\ -\ y)$

$\leq\ 4\,M\left[\frac{||x+y||^2}{4}\ +\ \frac{||x-y||^2}{4}\right] = 2\,M\left[||x||^2\ +\ ||y||^2\right]$ this is the parallelogram law. Now, you take $|\alpha|\ =\ 1,\ \alpha\,(Ax,\ y)\ =\ |A(x,\ y)|$ and use αx instead of x above.

So, you get $|(Ax,\ y)|\ \leq \frac{M}{2}\left[||x||^2\ +\ ||y||^2\right]$. $Ax\ \neq\ 0$, set $y\ =\frac{||x||}{||Ax||}\ Ax$. If $Ax\ =\ 0$, there is nothing for us to do, because we are looking at the maximum value of $(Ax,\ x)$. Using $y\ =\frac{||x||}{||Ax||}\ Ax$ in the above inequality,

$||x||\ ||Ax||\ \ \leq \frac{M}{2}\left[||x||^2\ +\ ||x||^2\right] = M\ ||x||^2 \Rightarrow ||Ax||\ \ \leq\ M\ ||x|| \Rightarrow ||A||\ \ \leq\ M$.

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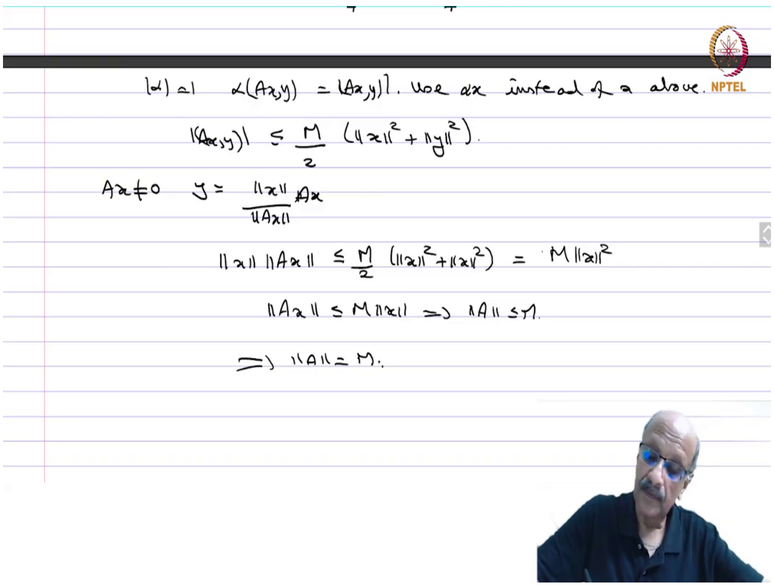

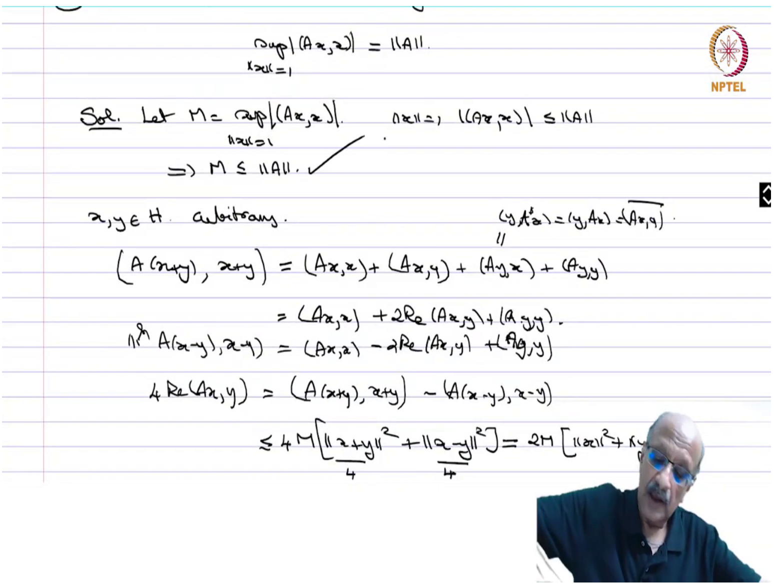

We already saw that $M \leq ||A||$ and therefore, we have $M = ||A||$ So, that proves.

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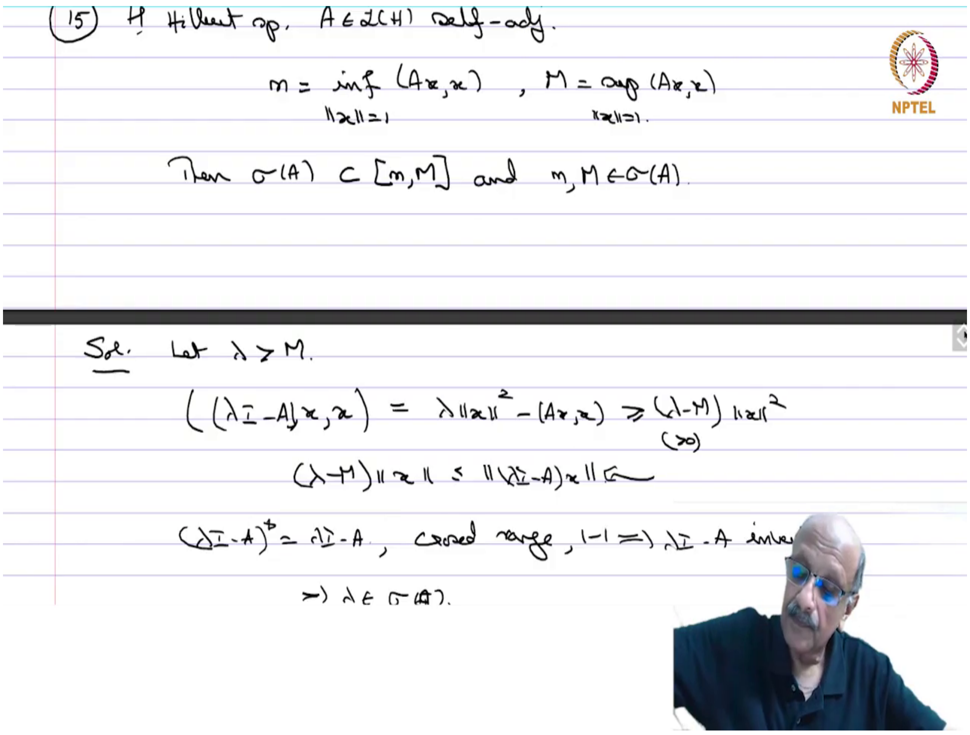

15. H, Hilbert space, $A \in L(H)$, self-adjoint. We already know that the spectrum is contained in the real line. Now, we are going to give some more precise information.

$m = \inf_{||x||=1}(Ax,\ x)$, and $M = \sup_{||x||=1} |(Ax,\ x)|$.

Then $\sigma(A) \subset [m,\ M]$ and $m,\ M \in \sigma(A)$.

So, we have a more precise characterization of the spectrum of a self-adjoint operator. **Solution.** Let $\lambda > M$. So, $((\lambda I - A)x,\ x) = \lambda||x||^2 - (Ax,\ x) \geq (\lambda - M)||x||^2$

and $(\lambda - M) > 0$. Now if you apply Cauchy Schwarz on the left-hand side, so, you will get $(\lambda - M)||x|| \leq ||(\lambda I - A)x||$. This is a familiar thing we have used many times in the theorem.

Lemma: $(\lambda I - A)^* = (\lambda I - A)$ and has closed range and 1-1, therefore, you have that $(\lambda I - A)$ is invertible. Therefore, $\lambda \in \rho(A)$.

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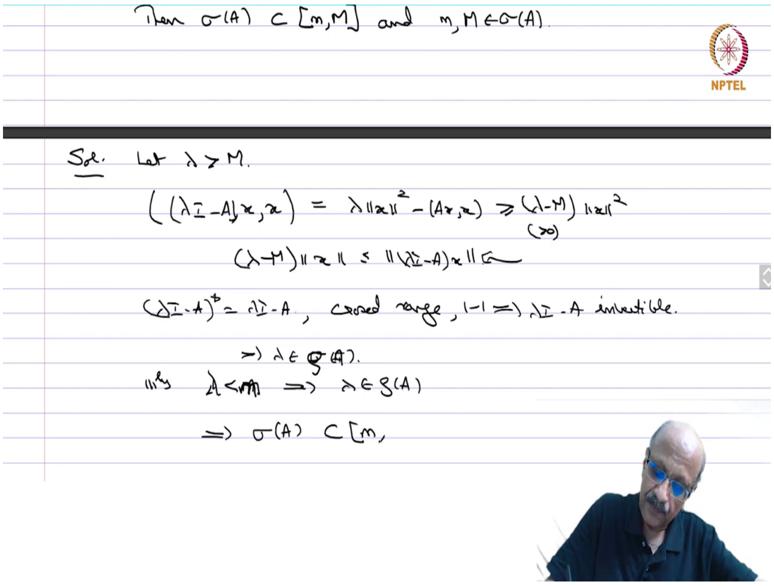

Similarly, for $\lambda < m$ you can do the same argument and you will get $\lambda < m \Rightarrow \lambda \in \rho(A)$.

Therefore, this implies that $\sigma(A) \subset [m, M]$.

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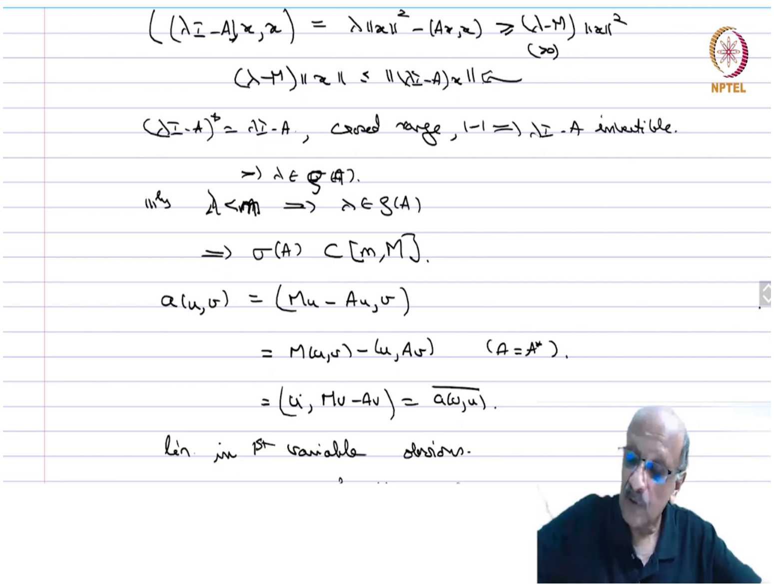

Now, you consider the bi linear form $a(u, v) = (M u - Au, v) = M(u, v) - (Au, v)$.

$(Au, v) = (u, A^* v) = (u, Av)$.

Thus, $a(u, v) = (M u - Au, v) = M(u, v) - (u, Av) = (u, Mv - Av) = \overline{a(u, v)}$.

We have this and of course, linear in the first variable that is obvious.

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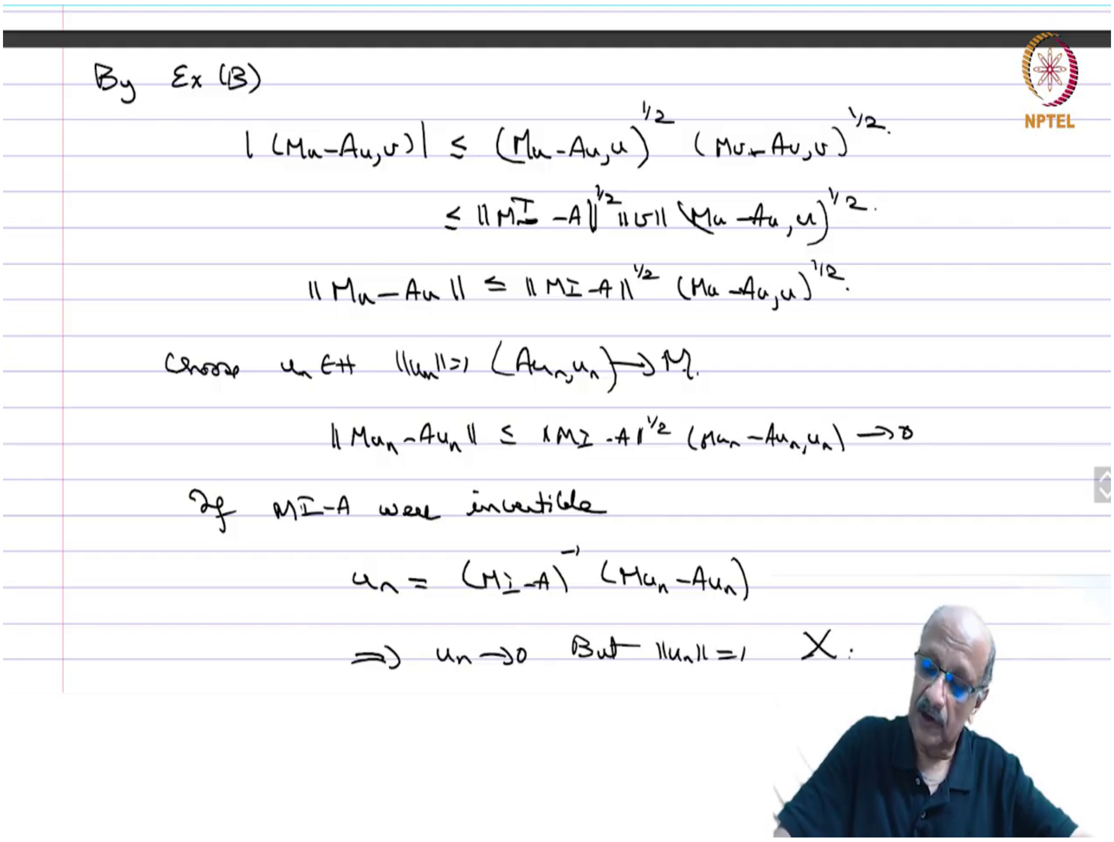

Therefore, by (exercise 13) we have the Cauchy Schwarz inequality.

$|(Mu - Au, v)| \leq (Mu - Au, u)^{1/2} (Mv - Av, v)^{1/2}$.

$$\leq ||M I - A||^{1/2} ||v|| (Mu - Au, u)^{1/2}.$$

Therefore, from this you get $||Mu - Au|| \leq ||M I - A||^{1/2} (Mu - Au, u)^{1/2}$.

Now, you choose $u_n \in H \quad ||u_n|| = 1, \ (Au_n, u_n) \rightarrow M$, this is supremum so, you can always find such a sequence. So, if you apply that you get

$||Mu_n - Au_n|| \leq ||M I - A||^{1/2} (Mu_n - Au_n, u_n)^{1/2} \rightarrow 0$

So, if $M I - A$ were invertible u_n can be written as $u_n = (M I - A)^{-1}(Mu_n - Au_n)$. I am just applying the operator and its inverse. So, I get back the identity. This implies that $u_n \rightarrow 0$, but $||u_n|| = 1$. So, we have a contradiction.

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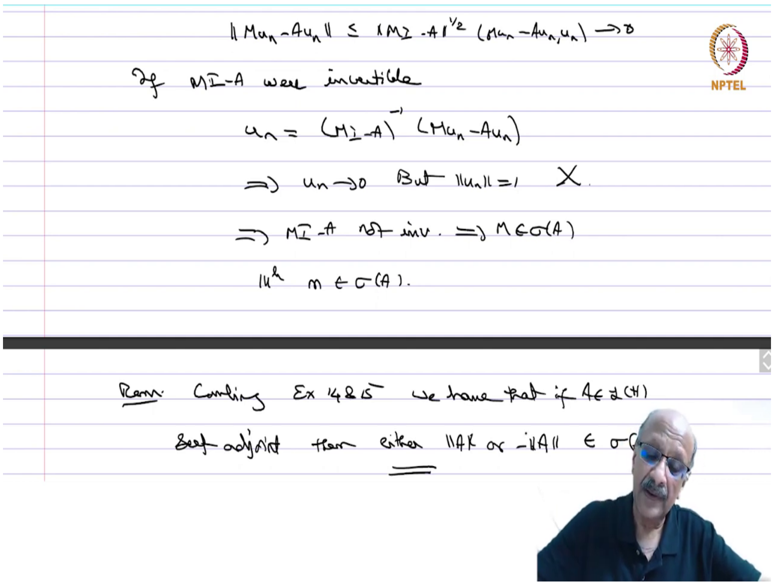

And therefore, this implies that $M\,I - A$ is not invertible $\Rightarrow\; M \in \sigma(A)$.

Similarly, you can show $m \in \sigma(A)$. So, that proves this thing.

Remark Combining exercise 13, 14 and 15 we have that if $A \in L(H)$, self-adjoint,

then either $||A||$ or $-\,||A|| \in \sigma(A)$.

Because either $||A|| = M$, or $-\,||A|| = m$. We do not know how the thing behaves, and therefore, one of them definitely has to be there, because $M = \sup|(Au, u)|$

So, you will get either m or M when you remove the modulus and therefore, you get one of them has to be in σ. so, we will wind up this chapter. And we will take up another topic next.