

Functional Analysis
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No. 62

Exercises – Part 3


(Refer Slide Time: 00:16)

(12) (a) H Hilbert sp. $S, T \in \mathcal{L}(H)$. Let $\lambda \neq 0$ be an eigenvalue of ST with eigenvector u . Show that Tu is an eigenvector of TS for eigenvalue λ .

Sol. $STu = \lambda u$, $\lambda \neq 0, u \neq 0 \Rightarrow \underline{Tu \neq 0}$.

$TS(Tu) = T(STu) = T(\lambda u) = \lambda T(u)$. $\Rightarrow Tu$ eigenvect. for TS corr. to λ .

(b)



Before we continue with the exercises is a small correction. **Exercise 7c**. We showed that $\alpha \|u - w\|^2 \leq a(u - w, u - v) + a(u - w, v - w)$ for all $v \in W$ and this since $v - w \in W$, $a(u - w, v - w) = 0$. So, $\alpha \|u - w\|^2 \leq M \|u - w\| \|u - v\|$ and therefore, you get that $\|u - w\| \leq \frac{M}{\alpha} \inf_{v \in W} \|u - v\|$.

12. (a) H Hilbert space and $S, T \in L(H)$. Let $\lambda \neq 0$ be an eigenvalue of ST with eigenvector u . Show that Tu is an eigenvector of TS for eigenvalue λ .

Solution. You have $STu = S(Tu) = \lambda u$, $\lambda \neq 0$, and obviously $u \neq 0$. So, this implies that Tu cannot be 0. So, $Tu \neq 0$. Now, $TS(Tu) = T(STu) = T(\lambda u) = \lambda T(u)$. Therefore, Tu is an eigenvector of TS for eigenvalue λ . So, that is a very trivial statement.

(Refer Slide Time: 03:37)

Sol. $STu = \lambda u$, $\lambda \neq 0, u \neq 0 \Rightarrow Tu \neq 0$.

$TS(Tu) = T(\lambda u) = \lambda Tu \Rightarrow Tu$ eigenvect. for TS corr. to λ .

(b) Non-zero eigenvalues of ST & TS are the same.

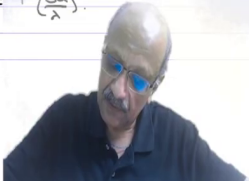
$T(N(ST - \lambda I)) = N(TS - \lambda I)$.

Sol. By (a) we can also show $\lambda \neq 0$ e.v. of $TS \Rightarrow \lambda$ e.v. of ST .

By (a) $T(N(ST - \lambda I)) \subset N(TS - \lambda I)$.

$u \in N(TS - \lambda I)$, $TSu = \lambda u \Rightarrow u = T\left(\frac{Su}{\lambda}\right)$.

Enough to show $\frac{Su}{\lambda}$ eigenvect of ST .



(b) Non-zero eigenvalues of ST and TS are the same and $T(N(ST - \lambda I)) = N(TS - \lambda I)$.

Solution. By (a) we can also show $\lambda \neq 0$ eigenvalue of $TS \Rightarrow \lambda$ eigenvalue of ST . So, that is the same thing as T and S is no nothing sacred about that. By (a) again if u is an eigenvector of ST , then Tu is eigenvector of TS and therefore, we have

$T(N(ST - \lambda I)) \subset N(TS - \lambda I)$. Now we have to show the converse. $u \in N(TS - \lambda I)$, we have $TSu = \lambda u \Rightarrow u = T\left(\frac{Su}{\lambda}\right)$. Now it is enough to show $\frac{Su}{\lambda}$ is an eigenvector of ST

(Refer Slide Time: 05:49)

$T(N(ST - \lambda I)) = N(TS - \lambda I)$.

Sol. By (a) we can also show $\lambda \neq 0$ e.v. of $TS \Rightarrow \lambda$ e.v. of ST .

By (a) $T(N(ST - \lambda I)) \subset N(TS - \lambda I)$.


$u \in N(TS - \lambda I)$, $TSu = \lambda u \Rightarrow u = T\left(\frac{Su}{\lambda}\right)$.

Enough to show $\frac{Su}{\lambda}$ eigenvect of ST .

$ST\left(\frac{Su}{\lambda}\right) = Su = \lambda\left(\frac{Su}{\lambda}\right)$.

$N(TS - \lambda I) \subset T(N(ST - \lambda I))$.

$\underline{=}$



So, we have to show the $ST\left(\frac{Su}{\lambda}\right) = S\left(\frac{T Su}{\lambda}\right) = Su$, that completes the proof. Therefore, we have shown $N(TS - \lambda I) \subset T(N(ST - \lambda I))$. So, we have shown that all the non-zero eigenvalues of TS and ST are the same.

(Refer Slide Time: 06:26)

By (a) $T(N(ST - \lambda I)) \subset N(TS - \lambda I)$.

$u \in N(ST - \lambda I)$, $T Su = \lambda u \Rightarrow \lambda u = T\left(\frac{Su}{\lambda}\right)$.

Enough to show $\frac{Su}{\lambda}$ eigenvector of ST .

$ST\left(\frac{Su}{\lambda}\right) = Su = \lambda\left(\frac{Su}{\lambda}\right)$.

$N(TS - \lambda I) \subset T(N(ST - \lambda I))$.

Rem. 1, No thing can be said about $\lambda = 0$ in inf. dim.

$H = l_2$ $Tx = (0, x_1, x_2, \dots)$ $x = (x_1, x_2, \dots)$

$Sx = (x_2, x_3, \dots)$

$Se_1 = 0$ e_1 eigenvector for $\lambda = 0$ for TS .

$\Rightarrow TSe_1 = 0$ $ST - T \Rightarrow \lambda = 0$ Not an eigenvalue

Remark 1: nothing can be said about $\lambda = 0$ in infinite dimensions. For instance you take $H = l_2$ and $Tx = (0, x_1, x_2, \dots)$, $x = (x_1, x_2, \dots)$ and $Sx = (x_2, x_3, \dots)$. So, both of them are bounded linear operators. If you take e_1 , then $Se_1 = 0$, so, e_1 is an eigenvector for $\lambda = 0$ for TS , because this also implies $TSe_1 = 0$.

On the other hand if you take ST , this is the identity map, implies $\lambda = 0$ not an eigenvalue.

(Refer Slide Time: 07:44)

$S e_1 = 0$ is an eigenvector for $\lambda = 0$ for TS .
 $\Rightarrow TS e_1 = 0$
 $ST = I \Rightarrow \lambda = 0$ Not an eigenvalue.

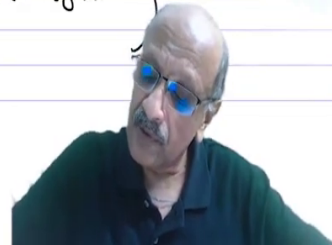


(ii) In finite dim. eigenvalues of ST & TS coincide.

$\lambda \neq 0$ already shown. $\lambda = 0$ eigenvalue of ST
 $\det(ST) = \det S \det T = 0 \Rightarrow TS$ is also singular
 $\Rightarrow \lambda = 0$ e.v. of TS .

(13) H , Hilbert space $\Rightarrow a(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$.
 s.t. $a(\alpha u + \beta v, z) = \alpha a(u, z) + \beta a(v, z)$.
 $a(u, v) = \overline{a(v, u)}$
 $a(u, u) \geq 0$

$\forall u, v, z \in H$.
 $\alpha, \beta \in \mathbb{C}$.




In finite dimensions we have eigenvalues of ST and TS coincide. We have already shown for $\lambda = 0$ you know already shown. So, if 0 is an eigenvalue of ST , means, $\det ST = \det S \det T = 0 \Rightarrow TS$ is also singular, because its determinant is also 0 implies $\lambda = 0$ is an eigenvalue of TS . In the finite dimensional case in fact you can show the characteristic polynomials are in fact the same. Now, so that you might have done in linear algebra.

13. H , Hilbert space, $a(\cdot, \cdot) : H \times H \rightarrow \mathbb{C}$, such that

$$a(\alpha u + \beta v, z) = \alpha a(u, z) + \beta a(v, z) \quad \forall u, v, z \in H \text{ and } \forall \alpha, \beta \in \mathbb{C}$$

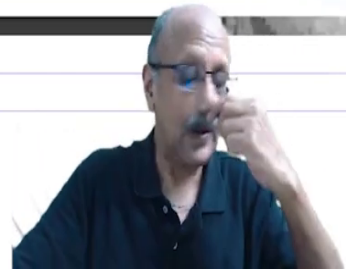
that is linear in the first variable. Then $a(u, v) = \overline{a(v, u)}$, and $a(u, u) \geq 0$. This is almost looking like an inner product except that the third condition is a bit weak.

(Refer Slide Time: 09:59)



$$a(u, u) \geq 0$$
$$|a(u, v)| \leq \sqrt{a(u, u)} \sqrt{a(v, v)} \quad \forall u, v \in H. \quad \checkmark$$

Sol. $u, v \in H \quad \alpha \in \mathbb{C} \quad |\alpha| = 1 \quad \alpha a(u, v) = |a(u, v)|.$

$$t \in \mathbb{R} \quad 0 \leq a(\alpha u - tv, \alpha u - tv)$$
$$= a(u, u) - t a(\alpha u, v) - t a(v, \alpha u) + t^2 a(v, v)$$
$$= a(u, u) - 2t \operatorname{Re} a(\alpha u, v) + t^2 a(v, v)$$
$$= a(u, u) - 2t |a(u, v)| + t^2 a(v, v)$$
$$\Rightarrow 4 |a(u, v)|^2 \leq 4 a(u, u) a(v, v)$$


It does not say that this is a norm in fact, $a(u, u) = 0$ may not mean that $u = 0$. But even then you have the analogue of the Cauchy Schwarz inequality, i.e.,

$|a(u, v)| \leq \sqrt{a(u, u)} \sqrt{a(v, v)}$, the square root is well defined for all $\forall u, v \in H$. So, this proof is identical as in the Cauchy Schwarz inequality.

Solution. You have $u, v \in H$ and then take $\alpha \in \mathbb{C}$, $|\alpha| = 1$, and $\alpha a(u, v) = |a(u, v)|$.

Let $t \in \mathbb{R}$. So, you have $0 \leq a(\alpha u - tv, \alpha u - tv)$ by the third property

$$= a(u, u) - t a(\alpha u, v) - t a(v, \alpha u) + t^2 a(v, v)$$

Now, by the second property $a(\alpha u, v) = \overline{a(v, \alpha u)}$. Therefore,

$$a(u, u) - t a(\alpha u, v) - t a(v, \alpha u) + t^2 a(v, v) = a(u, u) - 2t \operatorname{Re} a(\alpha u, v) + t^2 a(v, v)$$
$$= a(u, u) - 2t |a(u, v)| + t^2 a(v, v)$$

And now, you have a quadratic form which is never changed a sign that means the roots of the quadratic should either coincident or imaginary and therefore,

$$4 |a(v, v)|^2 \leq 4 a(u, u) a(v, v).$$

(Refer Slide Time: 12:49)

(14) H Hilbert sp. $A \in L(H)$, self-adj. Then

$$\sup_{\|x\|=1} |(Ax, x)| = \|A\|$$

Sol. Let $M = \sup_{\|x\|=1} |(Ax, x)|$. $\|x\|=1, |(Ax, x)| \leq \|A\|$
 $\Rightarrow M \leq \|A\|$.

$x, y \in H$ arbitrary. $(y, Ax) = \overline{(Ax, y)}$

$$(A(x+y), x+y) = (Ax, x) + (Ax, y) + (Ay, x) + (Ay, y)$$

$$= (Ax, x) + 2\operatorname{Re}(Ax, y) + (Ay, y)$$

$$\Rightarrow A(x+y), x+y = (Ax, x) + 2\operatorname{Re}(Ax, y) + (Ay, y)$$

$$4\operatorname{Re}(Ax, y) = (A(x+y), x+y) - A(x+y)$$

For the Cauchy Schwarz inequality, you do not need the full power of the inner product.

14. H , Hilbert space, $A \in L(H)$, self-adjoint. Then $\sup_{\|x\|=1} |(Ax, x)| = \|A\|$.

So, you have one more formula for the norm.

Solution. Let $M = \sup_{\|x\|=1} |(Ax, x)|$, if $\|x\| = 1$, then $|(Ax, x)| \leq \|A\| \|x\|^2 = \|A\|$.

Therefore, you have that $M \leq \|A\|$. Now, let $x, y \in H$ be arbitrary. So, let us compute

$(A(x + y), x + y) = (Ax, x) + (Ax, y) + (Ay, x) + (Ay, y)$. Now, what is (Ay, x) ?

$(Ay, x) = (y, A^* x) = (y, Ax) = \overline{(Ax, y)}$. And therefore,

$$(A(x + y), x + y) = (Ax, x) + (Ax, y) + (Ax, y) + (Ay, y)$$

$$= (Ax, x) + 2 \operatorname{Re}(Ax, y) + (Ay, y).$$

Similarly, $(A(x - y), x - y) = (Ax, x) - 2 \operatorname{Re}(Ax, y) + (Ay, y)$

So, when you subtract, you get $4 \operatorname{Re}(Ax, y) = (A(x + y), x + y) - (A(x - y), x - y)$.

(Refer Slide Time: 15:34)

$$= (Ax, x) + 2\operatorname{Re}(Ax, y) + (A, yy).$$

$$\operatorname{Re}(A(x-y), x-y) = (Ax, x) - 2\operatorname{Re}(Ax, y) + (Ay, y)$$

$$4 \operatorname{Re}(Ax, y) = (A(x+y), x+y) - (A(x-y), x-y)$$

$$\leq 4M \left[\frac{\|x+y\|^2}{4} + \frac{\|x-y\|^2}{4} \right] = 2M [\|x\|^2 + \|y\|^2]$$



$|\alpha| = 1 \quad \alpha(Ax, y) = |A(x, y)|$. Use αx instead of x above.

$$|A(x, y)| \leq \frac{M}{2} (\|x\|^2 + \|y\|^2).$$

$$Ax \neq 0 \quad y = \frac{\|x\|}{\|Ax\|} Ax$$

$$\|x\| \|Ax\| \leq \frac{M}{2} (\|x\|^2 + \|x\|^2) =$$



$$4 \operatorname{Re}(Ax, y) = (A(x + y), x + y) - (A(x - y), x - y)$$

$$\leq 4M \left[\frac{\|x+y\|^2}{4} + \frac{\|x-y\|^2}{4} \right] = 2M [\|x\|^2 + \|y\|^2] \text{ this is the parallelogram}$$

law. Now, you take $|\alpha| = 1$, $\alpha(Ax, y) = |A(x, y)|$ and use αx instead of x above.

So, you get $|A(x, y)| \leq \frac{M}{2} [\|x\|^2 + \|y\|^2]$. $Ax \neq 0$, set $y = \frac{\|x\|}{\|Ax\|} Ax$. If $Ax = 0$, there

is nothing for us to do, because we are looking at the maximum value of (Ax, x) . Using

$y = \frac{\|x\|}{\|Ax\|} Ax$ in the above inequality,

$$\|x\| \|Ax\| \leq \frac{M}{2} [\|x\|^2 + \|x\|^2] = M \|x\|^2 \Rightarrow \|Ax\| \leq M \|x\| \Rightarrow \|A\| \leq M.$$

(Refer Slide Time: 18:13)



$|x| = 1 \implies |(Ax, y) = (Ax, y)|$. Use ax instead of a above.

$$|(Ax, y)| \leq \frac{M}{2} (\|x\|^2 + \|y\|^2)$$

$$Ax \neq 0 \implies y = \frac{\|x\|}{\|Ax\|} Ax$$

$$\|x\| \|Ax\| \leq \frac{M}{2} (\|x\|^2 + \|x\|^2) = M \|x\|^2$$

$$\|Ax\| \leq M \|x\| \implies \|A\| \leq M$$

$$\implies \|A\| = M.$$



$$\sup_{\|x\|=1} |(Ax, x)| = \|A\|.$$

Sol. Let $M = \sup_{\|x\|=1} |(Ax, x)|$. $\|x\|=1, |(Ax, x)| \leq \|A\|$
 $\implies M \leq \|A\|$. ✓

$x, y \in H$. arbitrary.

$$(y, Ax) = (y, Ax) \overline{(Ax, y)}$$

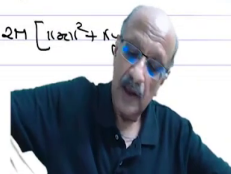
$$(A(x+y), x+y) = (Ax, x) + (Ax, y) + (Ay, x) + (Ay, y)$$

$$= (Ax, x) + 2\operatorname{Re}(Ax, y) + (Ay, y).$$

$$\operatorname{Re}(Ax, y) = \frac{1}{2} [(A(x+y), x+y) - (A(x-y), x-y)]$$

$$4 \operatorname{Re}(Ax, y) = (A(x+y), x+y) - (A(x-y), x-y)$$

$$\leq M \left[\frac{\|x+y\|^2}{4} + \frac{\|x-y\|^2}{4} \right] = 2M [\|x\|^2 + \|y\|^2]$$



We already saw that $M \leq \|A\|$ and therefore, we have $M = \|A\|$ So, that proves.

(Refer Slide Time: 18:39)

(15) H Hilbert sp, $A \in \mathcal{L}(H)$ self-adj.

$$m = \inf_{\|x\|=1} (Ax, x), \quad M = \sup_{\|x\|=1} (Ax, x)$$

Then $\sigma(A) \subset [m, M]$ and $m, M \in \sigma(A)$.



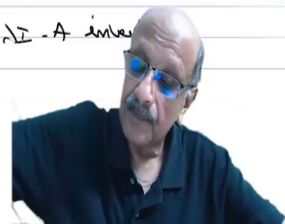
Sol. Let $\lambda > M$.

$$((\lambda I - A)x, x) = \lambda \|x\|^2 - (Ax, x) \geq (\lambda - M) \|x\|^2$$

$$(\lambda - M) \|x\| \leq \|(\lambda I - A)x\|$$

$(\lambda I - A)^* = \lambda I - A$, closed range, 1-1 $\Rightarrow \lambda I - A$ invertible

$\Rightarrow \lambda \in \rho(A)$.



15. H , Hilbert space, $A \in L(H)$, self-adjoint. We already know that the spectrum is contained in the real line. Now, we are going to give some more precise information.

$$m = \inf_{\|x\|=1} (Ax, x), \text{ and } M = \sup_{\|x\|=1} |(Ax, x)|.$$

Then $\sigma(A) \subset [m, M]$ and $m, M \in \sigma(A)$.

So, we have a more precise characterization of the spectrum of a self-adjoint operator.

Solution. Let $\lambda > M$. So, $((\lambda I - A)x, x) = \lambda \|x\|^2 - (Ax, x) \geq (\lambda - M) \|x\|^2$

and $(\lambda - M) > 0$. Now if you apply Cauchy Schwarz on the left-hand side, so, you will get $(\lambda - M) \|x\| \leq \|(\lambda I - A)x\|$. This is a familiar thing we have used many times in the theorem.

Lemma: $(\lambda I - A)^* = \lambda I - A$ and has closed range and 1-1, therefore, you have that $(\lambda I - A)$ is invertible. Therefore, $\lambda \in \rho(A)$.

(Refer Slide Time: 21:21)

Then $\sigma(A) \subset [m, M]$ and $m, M \in \sigma(A)$.

NPTEL

Sol. Let $\lambda > M$.

$$((\lambda I - A)x, x) = \lambda \|x\|^2 - (Ax, x) \geq (\lambda - M) \|x\|^2$$


$$(\lambda - M) \|x\|^2 \leq \|(\lambda I - A)x\|^2$$

$(\lambda I - A)^*$ is $\lambda I - A$, closed range, $\| \cdot \| \Rightarrow \lambda I - A$ invertible.

$\Rightarrow \lambda \in \rho(A)$.

$\forall \lambda < m \Rightarrow \lambda \in \rho(A)$

$\Rightarrow \sigma(A) \subset [m, M]$



Similarly, for $\lambda < m$ you can do the same argument and you will get $\lambda < m \Rightarrow \lambda \in \rho(A)$.

Therefore, this implies that $\sigma(A) \subset [m, M]$.

(Refer Slide Time: 21:48)

$$((\lambda I - A)x, x) = \lambda \|x\|^2 - (Ax, x) \geq (\lambda - M) \|x\|^2$$

$$(\lambda - M) \|x\|^2 \leq \|(\lambda I - A)x\|^2$$

NPTEL

$(\lambda I - A)^*$ is $\lambda I - A$, closed range, $\| \cdot \| \Rightarrow \lambda I - A$ invertible.

$\Rightarrow \lambda \in \rho(A)$.

$\forall \lambda < m \Rightarrow \lambda \in \rho(A)$

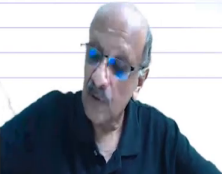
$\Rightarrow \sigma(A) \subset [m, M]$.

$$a(u, v) = (Mu - Au, v)$$

$$= M(u, v) - (u, Av) \quad (A = A^*)$$

$$= (u, Mv - Av) = \overline{a(u, v)}$$

lin in 1st variable obvious.



Now, you consider the bi linear form $a(u, v) = (Mu - Au, v) = M(u, v) - (Au, v)$.

$$(Au, v) = (u, A^* v) = (u, Av).$$

$$\text{Thus, } a(u, v) = (Mu - Au, v) = M(u, v) - (u, Av) = (u, Mv - Av) = \overline{a(u, v)}.$$

We have this and of course, linear in the first variable that is obvious.

(Refer Slide Time: 23:07)

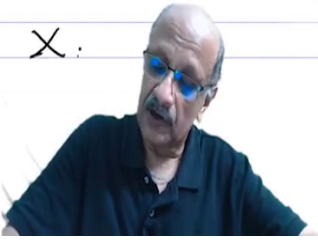

By Ex (B)

$$|(Mu - Au, v)| \leq (Mu - Au, u)^{1/2} (Mv - Av, v)^{1/2}$$
$$\leq \|MI - A\|^{1/2} \|v\| (Mu - Au, u)^{1/2}$$
$$\|Mu - Au\| \leq \|MI - A\|^{1/2} (Mu - Au, u)^{1/2}$$

Choose $u_n \in H$ ($\|u_n\| = 1$) $(Au_n, u_n) \rightarrow M$.

$$\|Mu_n - Au_n\| \leq \|MI - A\|^{1/2} (Mu_n - Au_n, u_n) \rightarrow 0$$

If $MI - A$ were invertible

$$u_n = (MI - A)^{-1} (Mu_n - Au_n)$$
$$\Rightarrow u_n \rightarrow 0 \text{ But } \|u_n\| = 1 \quad \times.$$


Therefore, by (exercise 13) we have the Cauchy Schwarz inequality.

$$|(Mu - Au, v)| \leq (Mu - Au, u)^{1/2} (Mv - Av, v)^{1/2}$$
$$\leq \|MI - A\|^{1/2} \|v\| (Mu - Au, u)^{1/2}$$


Therefore, from this you get $\|Mu - Au\| \leq \|MI - A\|^{1/2} (Mu - Au, u)^{1/2}$.

Now, you choose $u_n \in H$ $\|u_n\| = 1$, $(Au_n, u_n) \rightarrow M$, this is supremum so, you can always find such a sequence. So, if you apply that you get

$$\|Mu_n - Au_n\| \leq \|MI - A\|^{1/2} (Mu_n - Au_n, u_n)^{1/2} \rightarrow 0$$

So, if $MI - A$ were invertible u_n can be written as $u_n = (MI - A)^{-1} (Mu_n - Au_n)$. I am just applying the operator and its inverse. So, I get back the identity. This implies that $u_n \rightarrow 0$, but $\|u_n\| = 1$. So, we have a contradiction.

(Refer Slide Time: 25:28)



$$\|Mu_n - Au_n\| \leq \|M - A\|^{1/2} (Mu_n - Au_n) \rightarrow 0$$

of $M - A$ were invertible

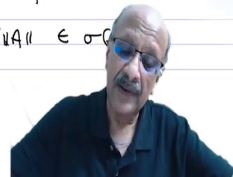
$$u_n = (M - A)^{-1} (Mu_n - Au_n)$$

$$\Rightarrow u_n \rightarrow 0 \text{ But } \|u_n\| = 1 \quad \times$$

$$\Rightarrow M - A \text{ not inv.} \Rightarrow M \in \sigma(A)$$

$$\mathbb{R}^n \quad m \in \sigma(A).$$

Remark Combining Ex 14 & 15 we have that if $A \in L(H)$ self adjoint then either $\|A\|$ or $-\|A\| \in \sigma(A)$



And therefore, this implies that $M I - A$ is not invertible $\Rightarrow M \in \sigma(A)$.

Similarly, you can show $m \in \sigma(A)$. So, that proves this thing.

Remark Combining exercise 13, 14 and 15 we have that if $A \in L(H)$, self-adjoint, then either $\|A\|$ or $-\|A\| \in \sigma(A)$.

Because either $\|A\| = M$, or $-\|A\| = m$. We do not know how the thing behaves, and therefore, one of them definitely has to be there, because $M = \sup |(Au, u)|$

So, you will get either m or M when you remove the modulus and therefore, you get one of them has to be in σ . so, we will wind up this chapter. And we will take up another topic next.